Concourse 18.02 Problem Set 11 – Fall 2018

due Fri, Dec 7

Read sections:

15.6 (The Divergence Theorem)
Notes V10 (The Divergence Theorem)

15.7 and **Notes V13** (Stokes' Theorem)

Notes V14 (Some Topological Questions)

Notes V15 (Relations to Physics)

1. [15.6/13] Use the Divergence Theorem to evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ where $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + 3\mathbf{k}$ and S is the boundary of the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$ and where **n** is the unit outward normal to the surface.

2. Verify the Divergence Theorem when S is the closed surface having for its sides a portion of the cylinder $x^2 + y^2 = 4$ and for its top and bottom circular portions of the planes $z = 0$ and $z = 3$; take **F** to be $\mathbf{F} = x^2\,\mathbf{i} + xy\,\mathbf{j} + z^2\mathbf{k}$.

3. [15.7/1] Use Stokes' Theorem to evaluate $\iint_S (\text{curl}\,\mathbf{F})\cdot\mathbf{n}\,dS$ where $\mathbf{F} = 3y\,\mathbf{i} - 2x\,\mathbf{j} + xyz\,\mathbf{k}$ and S is the hemispherical surface $z = \sqrt{4 - x^2 - y^2}$ with upward unit normal vector.

4. [15.7/9] Use Stokes' Theorem to evaluate $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds$ where $\mathbf{F} = \langle y - x, x - z, x - y\rangle$ and C is the boundary of the part of the plane $x + 2y + z = 2$ that lies in the first octant, oriented counterclockwise as viewed from above.

5. [SN-6F/1b] Verify Stokes' Theorem when S is the upper hemisphere of the sphere of radius one centered at the origin and C is its boundary; i.e., calculate both integrals in the theorem and show they are equal. Do this for the vector field $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$.

6. [SN-6F/2] Verify Stokes' Theorem if $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$ and S is the portion of the plane $x + y + z = 0$ cut out by the cylinder $x^2 + y^2 = 1$, and C is its boundary (an ellipse).

7. [SN-6F/5] Let S be the surface formed by the cylinder $x^2 + y^2 = a^2$, $0 \le z \le h$, together with the circular disc forming its top, oriented so the normal vector points up or out. Let $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j} + x^2\,\mathbf{k}$. Find the flux of $\nabla\times\mathbf{F}$ through S

   a) directly, by calculating two surface integrals;

   b) by using Stokes' Theorem.

8. [SN-6H/2] Show that for any closed surface S, and continuously differentiable vector field **F**,

$$\iint_S (\text{curl}\,\mathbf{F})\cdot d\mathbf{S} = 0.$$

   Do it two ways: a) using the Divergence Theorem; b) using Stokes' Theorem.

In problems 9 and 10, $\nabla^2 f = f_{xx} + f_{yy} + f_{zz}$ (the Laplacian of f)

9. Read the discussion in E&P §15.5 on heat flow in space (and in particular equations (22) and (23)). Then do E&P #25 in Chapter 15 Miscellaneous Exercises (at the end of the chapter and shown on next page).

Chap 15/#25. Deduce from the Divergence Theorem and eqn. (23) of Section 15.5 that the rate of heat flow across *S* into *B* is $R = \iiint_B K\,\nabla^2 u\,dV$.

[Reference: Suppose that a body has temperature $u = u(x,y,z)$ at the point (x,y,z). Experiments indicate that the flow of heat in the body is described by the heat-flow vector $\mathbf{q} = -K\,\overrightarrow{\nabla u}$. The number *K* – normally, but not always, a constant – is the heat conductivity of the body. The vector **q** points in the direction of heat flow, and its magnitude is the rate of flow of heat across a unit area normal to **q**. This flow rate is measured in units such as calories per second per square centimeter. If *S* is a closed surface within the body bounding the solid region *T* and **n** denotes the outer unit normal vector for *S*, then $\iint_S \mathbf{q}\cdot\mathbf{n}dS = -\iint_S K\,\overrightarrow{\nabla u}\cdot\mathbf{n}\,dS$ is the net rate of heat flow (in calories per second, for example) out of the region *T* across its boundary surface *S*.]

10. Suppose that the simple closed surface *S* is the iso-surface (level surface) of some smooth function $f(x,y,z)$, that is, the set of points in 3-space satisfying $f(x,y,z) = c$ for some constant *c*. Use the Divergence Theorem to show that if *G* is the interior of *S*, then $\iint_S \|\nabla f\|\,dS = \pm\iiint_G \nabla^2 f\,dV$.

11. Let $\mathbf{F}(x,y,z) = \left(\dfrac{-z}{x^2+z^2}\right)\mathbf{i} + y\,\mathbf{j} + \left(\dfrac{x}{x^2+z^2}\right)\mathbf{k}$ defined for all points (x,y,z) in 3-space not on the *y*-axis (that is, all points for which $x^2+z^2>0$.

a) By direct computation, show that $\nabla\times\mathbf{F} = \mathbf{0}$ for all points not on the *y*-axis.

b) By direct computation, show that $\oint_{C_1} \mathbf{F}\cdot d\mathbf{r} = 0$ where C_1 is the closed curve defined by $x^2+y^2=1,\ z=1$.

c) Can you use Stokes' Theorem and the fact (from part (a)) that $\nabla\times\mathbf{F} = \mathbf{0}$ to conclude that $\oint_{C_2} \mathbf{F}\cdot d\mathbf{r} = 0$ when C_2 is the closed curve defined by $x^2+z^2=1,\ y=0$? Why/ why not ?

d) Compute out $\oint_{C_2} \mathbf{F}\cdot d\mathbf{r} = 0$ and see what happens.