Concourse 18.02 Problem Set 10 – Fall 2011

Not to be turned in, but definitely do them!

Read sections:
15.7 and **Notes V13** (Stokes' Theorem)
Notes V14 (Some Topological Questions)
Notes V15 (Relations to Physics)

Part I Problems

15.7/1. Use Stokes' Theorem to evaluate $\iint_S (\text{curl}\,\mathbf{F})\cdot\mathbf{n}\,dS$ where $\mathbf{F} = 3y\,\mathbf{i} - 2x\,\mathbf{j} + xyz\,\mathbf{k}$ and S is the hemispherical surface $z = \sqrt{4 - x^2 - y^2}$ with upward unit normal vector.

15.7/9. Use Stokes' Theorem to evaluate $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds$ where $\mathbf{F} = \langle y - x, x - z, x - y\rangle$ and C is the boundary of the part of the plane $x + 2y + z = 2$ that lies in the first octant, oriented counterclockwise as viewed from above.

SN-6F/1b. Verify Stokes' Theorem when S is the upper hemisphere of the sphere of radius one centered at the origin and C is its boundary; i.e., calculate both integrals in the theorem and show they are equal. Do this for the vector field $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$.

SN-6F/2. Verify Stokes' Theorem if $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$ and S is the portion of the plane $x + y + z = 0$ cut out by the cylinder $x^2 + y^2 = 1$, and C is its boundary (an ellipse).

SN-6F/3. Verify Stokes' Theorem when S is the rectangle with vertices at $(0,0,0)$, $(1,1,0)$, $(0,0,1)$, and $(1,1,1)$, and $\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$.

SN-6F/5. Let S be the surface formed by the cylinder $x^2 + y^2 = a^2$, $0 \le z \le h$, together with the circular disc forming its top, oriented so the normal vector points up or out. Let $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j} + x^2\,\mathbf{k}$. Find the flux of $\nabla\times\mathbf{F}$ through S

(a) directly, by calculating two surface integrals;

(b) by using Stokes' Theorem.

SN-6G/1. Which regions are simply-connected?

a) first octant b) exterior of a torus c) region between two concentric spheres

d) three-space with one of the following removed:

i) a line ii) a point iii) a circle iv) the letter H v) the letter R vi) a ray

SN-6H/2. Show that for any closed surface S, and continuously differentiable vector field $\mathbf{F}$,

$$\iint_S (\text{curl}\,\mathbf{F})\cdot d\mathbf{S} = 0.$$

Do it two ways: a) using the Divergence Theorem; b) using Stokes' Theorem.

Part II problems on next page

Part II Problems

Problem 1: Let $\mathbf{F}(x,y,z)=\left(\frac{-z}{x^2+z^2}\right)\mathbf{i}+y\,\mathbf{j}+\left(\frac{x}{x^2+z^2}\right)\mathbf{k}$ defined for all points (x,y,z) in 3-space not on the *y*-axis (that is, all points for which $x^2+z^2>0$.

a) By direct computation, show that $\nabla\times\mathbf{F}=\mathbf{0}$ for all points not on the *y*-axis.

b) By direct computation, show that $\oint_{C_1}\mathbf{F}\cdot d\mathbf{r}=0$ where C_1 is the closed curve defined by

$x^2+y^2=1,\ \ z=1$.

c) Can you use Stokes' Theorem and the fact (from part (a)) that $\nabla\times\mathbf{F}=\mathbf{0}$ to conclude that $\oint_{C_2}\mathbf{F}\cdot d\mathbf{r}=0$ when C_2 is the closed curve defined by $x^2+z^2=1,\ \ y=0$? Why/ why not ?

d) Compute out $\oint_{C_2}\mathbf{F}\cdot d\mathbf{r}=0$ and see what happens.

Problem 2: Suppose the field **F** is replaced by the field

$\mathbf{G}(x,y,z)=\left(\frac{x}{x^2+y^2+z^2}\right)\mathbf{i}+\left(\frac{y}{x^2+y^2+z^2}\right)\mathbf{j}+\left(\frac{z}{x^2+y^2+z^2}\right)\mathbf{k}$ defined for all points $(x,y,z)\neq(0,0,0)$.

a) Show that $\nabla\times\mathbf{G}=\mathbf{0}$ for all points $(x,y,z)\neq(0,0,0)$.

b) Can you use Stokes' Theorem in this case and the fact that $\nabla\times\mathbf{G}=\mathbf{0}$ for all points $(x,y,z)\neq(0,0,0)$ to conclude that $\oint_{C}\mathbf{F}\cdot d\mathbf{r}=0$ for all simple closed curves *C* which do not pass through the origin?

c) Explain the difference between these two cases in term of the connectedness type of the domains of definition of the two fields.