Alternating Series; Absolute vs. Conditional Convergence; Ratio Test; Strategies

An **alternating series** is a series of the form $\sum_{n=1}^{\infty}(-1)^n b_n = -b_1 + b_2 - b_3 + b_4 - \cdots$ or $\sum_{n=1}^{\infty}(-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$ where $b_n > 0$ for all *n*.

The **Alternating Series Test** simply says that if the size of the terms of an alternating series are decreasing to zero, then the series will converge. The main idea of the Alternating Series Test is that if you are alternatively adding and subtracting smaller and smaller amounts and if these amounts eventually tend to zero, then the series will eventually narrow down to a finite sum.

Alternating Series Test: Let $\sum_{n=1}^{\infty}(-1)^n b_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1} b_n$ be an alternating series (where $b_n > 0$ for all *n*). If the terms are decreasing in size ($b_{n+1} < b_n$ for all *n*) and if $\lim_{n\to\infty} b_n = 0$, then the alternating series converges.

Estimating Sums

If we denote the sum of a convergent series by *S*, then if we truncate the series after a finite number of terms we will get the **partial sum** S_n. The sum of the infinitely many terms that we omit is called the **remainder** R_n.

Alternating Series Remainder Estimate: Suppose $\sum_{n=1}^{\infty}(-1)^n b_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1} b_n$ is an alternating series with decreasing terms and that $\lim_{n\to\infty} b_n = 0$. Then $|R_n| = |S - S_n| < b_{n+1}$.

This is true because the eventual sum *S* will always lie between the *n*th and (*n*+1)st partial sums, i.e. $|S - S_n| < |S_{n+1} - S_n| = b_{n+1}$. [Draw a picture of this on a number line and this should become clear.]

Ratio Test for Absolute Convergence: Given any series $\sum a_n$, consider the absolute series $\sum |a_n|$ and let $R = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$. [This may be thought of as the "eventual ratio."] If this limit exists, then

(1) If $R < 1$, then the series is **absolutely convergent**, i.e. both $\sum a_n$ and $\sum |a_n|$ converge.

(2) If $R > 1$, then the series is **divergent**, i.e. both $\sum a_n$ and $\sum |a_n|$ diverge.

(3) If $R = 1$, then the test is **inconclusive**. It's possible that the series could be either divergent, conditionally convergent, of absolutely convergent.

Note: A series $\sum a_n$ is conditionally convergent if it is convergent but its absolute series $\sum |a_n|$ is divergent.

Sample Problems

1. Determine the convergence of $\sum_{n=1}^{\infty}(-1)^n \operatorname{arccot}(n)$.

2. Determine the convergence of $\sum_{n=1}^{\infty}\frac{(-1)^n 2^n}{n^2}$.

3. Determine the convergence of $\sum_{n=1}^{\infty}(-1)^n \left|\sin(\tfrac{1}{n})\right|$

4. Determine the convergence of $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}$

5. Determine the convergence of $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2+2^n}$.

6. Determine the convergence of $\sum_{n=1}^{\infty}\frac{(-7)^n}{n}$.

7. Determine the convergence of $\sum_{n=1}^{\infty}(-1)^n \frac{3n-1}{2n+1}$

8. Estimate $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$ to within 0.001.

9. Show that the series $\sum_{n=1}^{\infty}\frac{(-2)^n}{n!}$ converges. How many terms of the series do we need in order to estimate the sum of the series with $|\text{error}| \le 0.01$.

10. Suppose $a_n \ge 0$ for all $n \ge 1$. Is it possible for $\sum_{n=1}^{\infty} a_n$ to converge conditionally?

11. For what values of p does the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n^p}$ converge absolutely?

12. Determine the convergence of $\sum_{n=1}^{\infty}\frac{n!}{(2n)!}$.

13. Determine the convergence of $1-\frac{1}{2^3}+\frac{1}{3^2}-\frac{1}{4^3}+\frac{1}{5^2}-\frac{1}{6^3}+\frac{1}{7^2}-\frac{1}{8^3}+\cdots$.

14. Determine the convergence of $\sum_{n=1}^{\infty}\frac{n!}{(n+2)!}$.

15. Determine the convergence of $\sum_{n=2}^{\infty}\frac{1}{\sqrt[3]{n^2-1}}$.

16. Determine the convergence of $\sum_{n=1}^{\infty}\frac{\arctan(n)}{1+n^2}$.

17. Determine the convergence of $\sum_{n=1}^{\infty}\frac{n^3}{n^5+4}$.

18. Determine the convergence of $\sum_{n=1}^{\infty}\frac{1}{n\,3^n}$.

19. Determine the convergence of $1+\frac{1}{1\cdot 3}+\frac{1}{1\cdot 3\cdot 5}+\frac{1}{1\cdot 3\cdot 5\cdot 7}+\cdots$.

Solutions:

1. The graph of $\operatorname{arccot}(x)=\cot^{-1}(x)$ shows this to be a decreasing function which tends to zero as x tends to infinity. This is all you need to show convergence for an alternating series, so this series is **convergent**.

2. For the series $\sum_{n=1}^{\infty}\frac{(-1)^n 2^n}{n^2}$, we see that the size of the nth term is $b_n=\frac{2^n}{n^2}$. If you look at the ratio $\frac{b_{n+1}}{b_n}=\frac{2^{n+1}n^2}{(n+1)^2 2^n}=\frac{2n^2}{(n+1)^2}\geq 2$ for all $n\geq 2$, so $b_{n+1}\geq 2b_n$ for all $n\geq 2$, and these terms are obviously growing bigger with n, so the series must be **divergent**. Also, by the Ratio Test for absolute convergence, we have $R=\lim_{n\to\infty}\left(\frac{b_{n+1}}{b_n}\right)=2>1$, so the series is divergent.

3. For the series $\sum_{n=1}^{\infty}(-1)^n\left|\sin(\frac{1}{n})\right|$, the limit $\lim_{n\to\infty}a_n=\lim_{n\to\infty}\left|\sin(\frac{1}{n})\right|$ does not exist, so the series is **divergent**.

4. This is an alternating p-series with $p=1/2$. The terms are decreasing to zero, so the alternating series converges. The absolute series is divergent, so the series is **conditionally convergent**.

5. For this series, consider its absolute series and use the Comparison Test with the (larger) series $\sum_{n=1}^{\infty}\frac{1}{2^n}$. This is a convergent geometric series, so the given series is **absolutely convergent**.

6. If you consider the absolute series, an easy calculation will show that the terms are not only not tending toward zero, they are tending toward infinity, so the series diverges (quickly).

7. For the series $\sum_{n=1}^{\infty}(-1)^n\frac{3n-1}{2n+1}$, the absolute value of the terms tend toward $3/2$. By the Ration Test, the series is **divergent**.

8. $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\cdots=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$. By the Alternating Series Remainder Estimate, since the terms are decreasing in size to zero, we need to have $|b_{n+1}|=\frac{1}{(n+1)^2}<.001=\frac{1}{1000}$. This will be true whenever $n+1>\sqrt{1000}$ or $n>\sqrt{1000}-1\cong 30.6$. So $n=31$ terms will be adequate to ensure this degree of accuracy.

9. First, use the Ratio Test to show that the series $\sum_{n=1}^{\infty}\frac{(-2)^n}{n!}$ is absolutely convergent. [The eventual ratio tends toward 0.] Therefore the terms will be decreasing after some point. [Actually, they'll be decreasing after the first two terms.] So we need to ensure that the $|b_{n+1}|\leq 0.01$. Trying a few terms, we quickly see that $n=7$ will be the first place where this becomes true, so 7 terms of this alternating series are required to achieve this degree of accuracy.

10. No. For a series with positive terms, either the series converges (absolutely) or it diverges.

11. For the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n^p}$, consider its absolute series $\sum_{n=1}^{\infty}\frac{\ln n}{n^p}$. Use the Integral Test to show that this series is convergent when $p>1$ and divergent when $p\le 1$. [Note: Calculating the integral is best done by integration by parts for the $p\ne 1$ case and by a substitution for the $p=1$ case.]

12. This one's an obvious candidate for the Ratio Test. Since $\left|a_n\right|=\frac{n!}{(2n)!}$, we calculate

$$\left|a_{n+1}\right|=\frac{(n+1)!}{(2(n+1))!}=\frac{(n+1)!}{(2n+2)!}=\frac{(n+1)n!}{(2n+2)(2n+1)(2n)!}$$

and $\left|\frac{a_{n+1}}{a_n}\right|=\frac{(n+1)n!(2n)!}{(2n+2)(2n+1)(2n)!n!}=\frac{1}{2(2n+1)}$.

Therefore $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{1}{2(2n+1)}=0<1$, so the absolute series converges and the series is **absolutely convergent**.

13. The series

$1-\frac{1}{2^3}+\frac{1}{3^2}-\frac{1}{4^3}+\frac{1}{5^2}-\frac{1}{6^3}+\frac{1}{7^2}-\frac{1}{8^3}+\cdots$ doesn't have a single expression for its *n*th term, but it is an alternating series, and its terms are approaching zero. However, they're not strictly decreasing in size so the Alternating Series Test is not applicable. However, the absolute series is the sum of two convergent series that are both comparable to *p*-series (with $p=2$ and $p=3$) with the terms intertwined. Specifically, the absolute series is

$$1+\frac{1}{2^3}+\frac{1}{3^2}+\frac{1}{4^3}+\frac{1}{5^2}+\frac{1}{6^3}+\frac{1}{7^2}+\frac{1}{8^3}+\cdots$$

$$=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$$

$$+\frac{1}{2^3}+\frac{1}{4^3}+\frac{1}{6^3}+\frac{1}{8^3}+\cdots$$

$$=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}+\sum_{n=1}^{\infty}\frac{1}{(2n)^3}$$

If we apply the Limit Comparison Test to both of them, it's easy to see that they both converge, so the original series is **absolutely convergent**.

14.

$$\sum_{n=1}^{\infty}\frac{n!}{(n+2)!}=\sum_{n=1}^{\infty}\frac{n!}{(n+2)(n+1)n!}=\sum_{n=1}^{\infty}\frac{1}{(n+2)(n+1)}$$

This series is comparable (using the Limit Comparison Test) to a *p*-series with $p=2$, so it's **absolutely convergent**.

15. Again, using the Limit Comparison Test, the series $\sum_{n=2}^{\infty}\frac{1}{\sqrt[3]{n^2-1}}$ is comparable to a *p*-series with $p=2/3<1$, so it's **divergent**.

16. For the series $\sum_{n=1}^{\infty}\frac{\arctan(n)}{1+n^2}$, note that all the terms are positive and that the numerator never exceeds $\pi/2$. Use the Comparison Test with the (larger) series $\sum_{n=1}^{\infty}\frac{\pi/2}{1+n^2}=\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{1+n^2}<\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}$. Since the latter series is a convergent *p*-series with $p=2$, the original series is **absolutely convergent**.

17. The series $\sum_{n=1}^{\infty}\frac{n^3}{n^5+4}$ is comparable (using the Limit Comparison Test) to a *p*-series with $p=2$, the given series is **absolutely convergent**.

18. The series $\sum_{n=1}^{\infty}\frac{1}{n\,3^n}$ is an obvious candidate for the Ratio Test because of the exponential function. We calculate

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{n\,3^n}{(n+1)3^{n+1}}=\lim_{n\to\infty}\frac{n}{(n+1)3}=\frac{1}{3}<1.$$

So the series is **absolutely convergent**.

19. For the series $1+\frac{1}{1\cdot 3}+\frac{1}{1\cdot 3\cdot 5}+\frac{1}{1\cdot 3\cdot 5\cdot 7}+\cdots$, we calculate the ratio of successive terms directly:

$$\left|\frac{a_2}{a_1}\right|=\frac{1}{3},\ \left|\frac{a_3}{a_2}\right|=\frac{1}{5},\ \left|\frac{a_4}{a_3}\right|=\frac{1}{7},\ \ldots\left|\frac{a_{n+1}}{a_n}\right|=\frac{1}{2n+1}.$$

Therefore $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{1}{2n+1}=0<1$, so the series is (absolutely) **convergent**.

Strategies for Determining Convergence or Divergence of a Series

1. If the series is of the form $\sum 1/n^p$, then it is a ***p*-series**. The series converges for $p > 1$ and diverges for $p \leq 1$.
2. If the series is of the form $\sum ar^n$ for some constant *r*, then it is a **geometric series**. It converges for $|r| < 1$ and diverges for $|r| \geq 1$.
3. If the series is similar to (comparable to) a *p*-series or a geometric series, consider using the **Comparison Test** or the **Limit Comparison Test**.
4. If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges (by the **Divergence Test**).
5. If the series is of the form $\sum_{n=1}^{\infty}(-1)^n b_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1} b_n$ where $b_n > 0$ for all *n*, consider using the **Alternating Series Test**. If the terms are decreasing in size ($b_{n+1} < b_n$ for all *n*) and if $\lim_{n\to\infty} b_n = 0$, then the alternating series converges. You might also consider using the **Ratio Test for Absolute Convergence**.
6. If the series involves products, factorial expressions, or exponential functions (constant raised to the *n*th power or a power that contains *n*), consider using the **Ratio Test**.
7. If the *n*th term of the series $a_n = f(n)$ is such that the integral $\int_1^{\infty} f(x)dx$ is relatively easy to evaluate, then you may want to consider using the **Integral Test**, assuming the hypotheses of the test are satisfied.
8. Is the series a **telescoping series** or can it be put in the form of a telescoping series after using partial fractions to re-express the *n*th term of the series? If so, convergence or divergence can be determined by directly computing an expression for the *n*th partial sum of the series and finding the limit of the partial sums of the series.

Additional Practice Problems

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^2+n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$
3. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$
4. $\displaystyle\sum_{k=1}^{\infty} \frac{5^k}{3^k+4^k}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$
6. $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$
7. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$
8. $\displaystyle\sum_{n=1}^{\infty} \tan(1/n)$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3-2n^2+5}$
10. $\displaystyle\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$
11. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+2}}\right)$