Concourse Math 18.02 Mega-List of Things You May Want to Know – Fall 2012

1. Write equations for surfaces in $\mathbf{R}^3$ that are characterized geometrically by distances (spheres, cylinders, etc.), and determine their intersection with specified planes.

2. Sketch or identify contour plots of real-valued functions on $\mathbf{R}^2$. Identify plots of function graphs for such functions.

3. Given vectors in $\mathbf{R}^2$ or $\mathbf{R}^3$, do addition, scalar multiplication, dot and cross products.

4. Manipulate vector expressions symbolically (distributive law, triple product, $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$).

5. Express lengths, angles, areas of triangles and parallelograms, and volumes of parallelepipeds and tetrahedra in terms of vectors. Identify and construct orthogonal vectors.

6. Find the scalar and vector projections of a vector in any given direction.

7. Calculate the intersection (if any) of specified lines and planes. Resolve a vector into components parallel and perpendicular to a specified vector, line, or plane.

8. Calculate the distance from a point to a specified line or plane, or between two nonintersecting lines in $\mathbf{R}^3$.

9. Calculate partial derivatives and directional derivatives of functions of two or more variables.

10. Calculate the gradient of a real-valued function of two or three variables, and state and apply the relation between gradients and level curves or surfaces.

11. Calculate the rate of change of a real-valued function along a parametrized path.

[Basic Chain Rule: $\frac{d}{dt}\left[f(x(t), y(t))\right] = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \nabla f \cdot \mathbf{v}$ for a path in $\mathbf{R}^2$;

$$\frac{d}{dt}\left[f(x(t), y(t), z(t))\right] = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \nabla f \cdot \mathbf{v} \text{ for a path in } \mathbf{R}^3.]$$

12. Given a surface in $\mathbf{R}^3$ described by the graph of a function $f(x, y)$, find the equation of the tangent plane at a point on the surface where f is known to be differentiable, and use it to find approximate values for this function near the point of tangency.

[Linear approximation: $f(x,y) \cong f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$]

13. Given a function $s = f(u, v)$ or $s = f(u, v, w)$ whose arguments u, v, w are specified functions of x, y, and perhaps z, use the chain rule to derive or verify relationships among the partial derivatives of s with respect to x, y, and z.

14. Given functions that express Cartesian coordinates x and y in terms of other coordinates u and v, and a function $z = f(x, y)$, use the chain rule to express partial derivatives of z with respect to u and v in terms of partial derivatives of f with respect to x and y.

15. Given a curve in the plane specified by $f(x, y)$ = constant, use implicit differentiation to find a formula for the derivative of the function that specifies y in terms of x near a specified point on the curve, and use the value of this derivative to determine a tangent line or to do a linear approximation near the point.

16. Given a surface specified by $f(x, y, z)$ = constant, use implicit differentiation to find formulas for the partial derivatives of the function that specifies z in terms of x and y near a specified point on the surface, and use the values of these derivatives to determine a tangent plane or to do a linear approximation near the point.

17. Given a contour diagram of a function $f(x, y)$, identify the signs of f_x, f_y, f_{xx}, f_{xy}, f_{yx}, and f_{yy} by examining the spacing of the contours and the values of f on the contours.

18. For a function of two variables, construct an approximating function near a specified point (x_0, y_0) that includes both linear and quadratic terms in $(x - x_0)$ and $(y - y_0)$.

19. Given a function of two or three variables, locate all its stationary points, or determine whether or not a specified point is a stationary point.

20. Write down the Hessian matrix for a function of two variables at a given stationary point and use it to determine whether the stationary point is a minimum, a maximum, or a saddle point. Identify cases where the Hessian cannot answer this question.

21. Sketch or identify level curves of a function of two variables in the vicinity of a stationary point.

22. Formulate and solve optimization problems that involve minimizing a sum of squares.

23. Use the method of Lagrange multipliers to find the stationary points of a function $f(x, y)$ of two variables subject to a constraint $g(x, y)$ = constant. Sketch or identify level curves and gradient vectors for f and g in the vicinity of one of these stationary points.

24. Use the method of Lagrange multipliers to find the stationary points of a function $f(x, y, z)$ of three variables subject to a constraint $g(x, y, z)$ = constant.

25. Given a region of the plane that includes its boundary, enumerate all the points that are candidates for the location of the maximum or minimum value of the function on that region.

26. Find the maximum and minimum values of a specified function on a region of the plane that is bounded by one, two, or three lines or curves.

27. Write down or identify a Riemann sum whose limit is the double or triple integral of a specified function over a specified region.

28. Given a region in the plane that can be divided into strips bounded by function graphs, express a double integral over the region as an iterated integral in Cartesian coordinates, and evaluate the iterated integral by using antiderivatives, if possible.

29. Given a region in $\mathbf{R}^3$ that is bounded by planes (not necessarily all perpendicular to the coordinate axes), express a triple integral over the region as an iterated integral in Cartesian coordinates, and evaluate the iterated integral by using antiderivatives, if possible.

30. Given an iterated double or triple integral, identify the domain of integration and express the integral as an iterated integral with a different order of integration.

31. Given a region in the plane that is bounded by circular arcs, radial lines, and graphs of functions expressing r in terms of θ, express a double integral over the region as an iterated integral in polar coordinates, and evaluate the iterated integral by using antiderivatives, if possible.

32. Given an iterated double or triple integral in one coordinate system, identify the domain of integration and express the integral in terms of a different coordinate system.

33. Given a region in $\mathbf{R}^3$ whose boundary consists of one or two planes perpendicular to the z-axis plus part of a cylinder, cone or sphere symmetrical about the z-axis, express a triple integral over the region as an iterated integral in cylindrical coordinates, and evaluate the iterated integral by using antiderivatives, if possible.

34. Given a region in $\mathbf{R}^3$ whose boundary consists of portions of one or two spheres, planes that contain the z-axis or that are perpendicular to the z-axis, and cones whose vertex is the origin and whose axis is the z-axis, express a triple integral over the region as an iterated integral in spherical coordinates, and evaluate the iterated integral by using antiderivatives, if possible.

35. Use multiple integrals to calculate the average value of a function of a region in $\mathbf{R}^2$ or $\mathbf{R}^3$, the mass of a region with a specified density function, the area or volume of a given region in $\mathbf{R}^2$ or $\mathbf{R}^3$, or the centroid (geometric center) of a region in $\mathbf{R}^2$ or $\mathbf{R}^3$ using appropriate coordinates.

36. Given a new coordinate system in the plane with coordinates *u* and *v* and functions $x(u, v)$ and $y(u, v)$ that express the usual Cartesian coordinates in terms of the new coordinates, along with a region of integration whose boundary can easily be expressed in terms of *u* and *v*, convert a double integral over the region into an iterated integral over *u* and *v*.

37. Given a new coordinate system in $\mathbf{R}^3$ with coordinates *u*, *v*, and *w* and functions $x(u, v, w)$, $y(u, v, w)$, and $z(u, v, w)$ that express the usual Cartesian coordinates in terms of the new coordinates, along with a region of integration whose boundary can easily be expressed in terms of *u*, *v*, and *w*, convert a triple integral over the region into an iterated integral over *u*, *v*, and *w*.

38. Given sufficient data, describe a line or plane either parametrically or in terms of one or more equations involving *x*, *y*, and *z*.

39. Write one or more parametrized expressions for a specified path in $\mathbf{R}^2$ or $\mathbf{R}^3$.

40. Sketch a graph of a parametrized path, and eliminate the parameter (if possible) to find an equation or equations in terms of *x*, *y*, (and *z*) for the path.

41. For motion in $\mathbf{R}^2$ or $\mathbf{R}^3$ specified parametrically in terms of time, calculate and sketch position, velocity, and acceleration vectors, and find an expression for the tangent line to the path at a specified point and for the unit tangent vector.

42. Solve problems based on motion of one or two particles described parametrically; e.g. when does it cross a specified plane, when is it closest to a specified point, what is the distance of closest approach of two moving particles?

43. For motion with specified constant acceleration and specified initial velocity and position, write a parametrized expression for the path.

44. Develop an expression for the length of a parametrized path as an integral.

45. Make or identify sketches of vector fields in $\mathbf{R}^2$.

46. Calculate the line integral of a specified vector field over a specified path, inventing a parametrization if necessary.

47. Given a curve that is easily parametrized, set up and evaluate an integral to determine the length of the curve, the integral of a density function over the curve, or the average value of a function over the curve.

48. Test whether a given vector field in $\mathbf{R}^2$ or $\mathbf{R}^3$ is conservative. For a conservative field, find a (potential) function of which it is the gradient.

49. Calculate the line integral of a conservative vector field using the Fundamental Theorem of Line Integrals, i.e. $\int_\gamma \nabla V \cdot d\mathbf{x} = V(\mathbf{x}(b)) - V(\mathbf{x}(a))$ for a path γ described by $\mathbf{x}(t)$ for $a \le t \le b$.

50. Given the line integral of a vector field over the boundary of a region in the plane, use Green's theorem to convert it to a double integral over the region.

51. Use Green's theorem for efficient calculation of area, center of mass, or moment of inertia for a region in the plane whose boundary consists of a small number of easily-parametrized curves.

52. Given two parameters *u* and *v*, intervals for *u* and *v*, and functions that express *x*, *y*, and *z* in terms of *u* and *v*, identify, describe, or sketch the surface in $\mathbf{R}^3$ that is specified by this parametrization. At a given point on the surface, find two independent vectors that are tangent to the surface and one that is normal to the surface.

53. Given a surface that is a portion of a plane, sphere, cylinder, or cone in $\mathbf{R}^3$, invent a parametrization $\mathbf{r}(u,v)$ for it, choosing parameters u and v so that the vector $\dfrac{\partial \mathbf{r}}{\partial u}\times\dfrac{\partial \mathbf{r}}{\partial v}$ points in a specified direction normal to the surface.
54. Given a surface that is easily parametrized, set up and evaluate a double integral to determine the area of the surface, the integral of a density function over the surface, or the average value of a function over the surface.
55. Given a surface that is easily parametrized, set up and evaluate an integral to determine the flux of a specified vector field through the surface.
56. Given a closed surface that consists of a small number of pieces each of which is easily parametrized (for example, a cylinder plus its top and bottom, or a cone plus its top, or a hemisphere plus a disk in its equatorial plane), evaluate the flux of a specified vector field out through this closed surface by forming the sum of appropriate flux integrals.
57. Given a vector field expressed in Cartesian coordinates, calculate the divergence or curl of the vector field.
58. Given the flux integral of a vector field over the boundary of a region in $\mathbf{R}^3$, use the divergence theorem to convert it to a triple integral over the region.
59. Given the line integral of a vector field around the boundary of a surface in $\mathbf{R}^3$, use Stokes' theorem to convert it to a flux integral over the surface.