Essential Topics of Math 18.01a

1) **Fundamental Theorem of Calculus, 1st Version**. The first of these can be formulated either as the Evaluation Theorem or the Net Change Theorem.

Evaluation Theorem: If *f* is continuous on the interval $[a,b]$, then $\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of *f*, that is, $F' = f$.

Net Change Theorem: The integral of a rate of change is the net change: $\int_a^b F'(t)dt = F(b) - F(a)$

Fundamental Theorem of Calculus, 2nd Version
The second of these Fundamental Theorems of Calculus says essentially that the derivative of a definite integral with a variable upper limit gives you back the original integrand. That is, if $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$. This is also sometimes expressed as $\frac{d}{dx}\int_a^x f(t)dt = f(x)$.

This version of the Fundamental Theorem can also be enhanced using the Chain Rules to allow functions of *x* in both the lower and upper integral limits. This is then known as **Leibnitz' Theorem**:

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f(h(x))h'(x) - f(g(x))g'(x)$$

2) **Integration Techniques**
The main techniques we covered were basic <u>substitutions</u>, <u>integration by parts</u>, integrating <u>powers and products of trigonometric functions</u>, <u>trigonometric substitutions</u>, and the method of <u>partial fractions</u> for integrating rational functions.

3) **Approximate Integration**
There were five methods discussed for how to approximate a definite integral by partitioning the domain and estimating the areas associated with the subintervals created in this way. The first two methods (using left-hand endpoints or using right-hand endpoints) do not give very good approximations, but the **Midpoint Rule** and the **Trapezoidal Rule** both give reasonably good approximations. The Midpoint Rule generally gives half the error (hence twice the accuracy) as the Trapezoidal Rule. The accuracy can be estimated by the following error bounds:

If the integrand $f(x)$ is such that $|f''(x)| \le K$ for $a \le x \le b$, then the <u>error bounds for the Trapezoidal Rule and Midpoint Rule</u> for estimating $\int_a^b f(x)dx$ will be, respectively, $|E_T| \le \frac{K(b-a)^3}{12n^2}$ and $|E_M| \le \frac{K(b-a)^3}{24n^2}$.

[<u>Note</u>: Simpson's Rule actually gives much greater accuracy that either the Midpoint Rule of the Trapezoidal Rule. If the integrand $f(x)$ is such that $|f^{(4)}(x)| \le K$ for $a \le x \le b$, then the error bound for the Simpson's Rule for estimating $\int_a^b f(x)dx$ will be $|E_S| \le \frac{K(b-a)^5}{180n^4}$.]

4) **Improper Integrals**:
There are two primary types of <u>improper integrals</u> (integrals which cannot be defined using the standard definition of a definite integral involving the method of Riemann Sums). They are a) integrals where the domain is not finite; and b) integrals where the integrand has an essential discontinuity either at one or both endpoints or within the domain of integration. In either case, we work around the problem by using limits to either approach infinity or to approach the discontinuity or discontinuities.

5) **Applications of integration in geometric measurement**

a) Area under a graph; area between two graphs. (Ref: section 6.1)
b) Volumes of solid regions with known cross-sections (Ref: section 6.2)
c) Volumes of solids of revolution using disks or washers (cross-sections perpendicular to the axis of rotation) or cylindrical shells (cross-sections parallel to the axis of rotation). (Ref: sections 6.2 and 6.3)
d) Arclength of the graph of a function
e) Surface area of a surface of revolution

6) **Average value of a function**

7) **Applications of integration in physics: Work**

Relevant problems covered were those involving lifting ropes and chains (with a linear density that was either given or which could be calculated); and those involving pumping of liquids into and out of various shaped containers (with a density in either kg/m^3 or lb/ft^3).

8) **Sequences**

You should understand the technical definition of the limit of a sequence and what it means to say that a sequence is convergent. You should be able to express a sequence of numbers either by a formula for the *n*th term of the sequence or by an appropriate recursion formula (where the *n*th term is expressed in terms of its predecessors in the sequence.

9) **Series**

a) Above all you must be clear about the distinction between a sequence (an infinite list of numbers) and a series (an infinite sum of numbers).
b) You should fully understand what it means to say that a series converges, i.e. that its corresponding sequence of partial sums is convergent.
c) You should be able to recognize a geometric series (including a repeating decimal), to know when it converges, and to calculate the sum of a convergent geometric series.
d) You should be familiar with some of the standard series that we regularly use for comparing with other series. These include geometric series and *p*-series (including the harmonic series).
e) You should know the Divergence Test, i.e. the fact that if the limit of the terms of any series do not approach zero, then the series cannot possibly converge. However, you should also be very clear about the fact that just because the individual terms of a series are approaching zero, this does not guarantee convergence.

10) **Series with positive terms – Convergence Tests**

The principal convergences tests that we use in this case are:
a) The Integral Test
b) The Comparison Test
c) The Limit Comparison Test
d) The Ratio Test

11) **Alternating Series**

When the terms of a series alternate in sign, convergence is generally much simpler to understand than in the case where all the terms are positive. You should know:
a) The definition of an alternating series
b) The Alternating Series Test
c) The Alternating Series Estimation Theorem

d) The distinction between an absolutely convergent series, a conditionally convergent series, and a divergent series and how to use the Ratio Test for Absolute Convergence.

12) **Power Series**

a) You should understand what a power series is and what it means to have a power series in x (or about 0 or centered at 0) or a power series in $(x-a)$ (or about a or centered at a).

b) You should be able to apply the Ratio Test for Absolute Convergence to determine the interval of convergence of a power series (or to show that it converges for all x).

c) You should be familiar with several familiar power series representations for the functions $\frac{1}{1-x}$, e^x, $\sin x$, and $\cos x$ (and their intervals of convergence) and how to manipulate these series to produce other power series representations for functions (and their intervals of convergence) using differentiation, integration, substitution, and ordinary algebraic operations.

13) **Taylor and Maclaurin Series**

a) You should know the definition of Taylor and Maclaurin polynomials and Taylor and Maclaurin series and how to calculate the coefficients.

b) You should know how to manipulate a known Taylor or Maclaurin series to produce the corresponding series representation for related functions.

c) You should know how to use a series representation of a function to approximate the integral of a function that might otherwise be difficult or impossible to calculate using the Fundamental Theorem of Calculus.

d) You should be able to use either the Alternating Series Remainder Estimate or Taylor's Theorem (or Taylor's Inequality) to relate the number of terms of a Taylor polynomial approximation (n), the distance from the center of the approximation ($|x-a|$), and the error $|R_n(x)|$. (Usually, it's easier to use the Alternating Series Remainder Estimate.)