w

V9. Surface Integrals

Surface integrals are a natural generalization of line integrals: instead of integrating over a curve, we integrate over a surface in 3-space. Such integrals are important in any of the subjects that deal with continuous media (solids, fluids, gases), as well as subjects that deal with force fields, like electromagnetic or gravitational fields.

Though most of our work will be spent seeing how surface integrals can be calculated and what they are used for, we first want to indicate briefly how they are defined. The surface integral of the (continuous) function $f(x, y, z)$ over the surface S is denoted by

$$\iint_S f(x,y,z)\, dS \ . \tag{1}$$

You can think of dS as the area of an infinitesimal piece of the surface S. To define the integral (1), we subdivide the surface S into small pieces having area ΔS_i, pick a point (x_i, y_i, z_i) in the i-th piece, and form the Riemann sum

$$\sum f(x_i, y_i, z_i)\Delta S_i \ . \tag{2}$$

w As the subdivision of S gets finer and finer, the corresponding sums (2) approach a limit which does not depend on the choice of the points or how the surface was subdivided. The surface integral (1) is defined to be this limit. (The surface has to be smooth and not infinite in extent, and the subdivisions have to be made reasonably, otherwise the limit may not exist, or it may not be unique.)

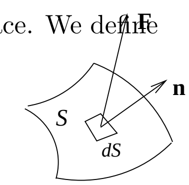
1. The surface integral for flux.

The most important type of surface integral is the one which calculates the flux of a vector field across S. Earlier, we calculated the flux of a plane vector field $\mathbf{F}(x, y)$ across a directed curve in the xy-plane. What we are doing now is the analog of this in space.

We assume that S is *oriented*: this means that S has two sides and one of them has been designated to be the *positive side*. At each point of S there are two unit normal vectors, pointing in opposite directions; the *positively directed* unit normal vector, denoted by $\mathbf{n}$, is the one standing with its base (i.e., tail) on the positive side. If S is a closed surface, like a sphere or cube — that is, a surface with no boundaries, so that it completely encloses a portion of 3-space — then by convention it is oriented so that the outer side is the positive one, i.e., so that $\mathbf{n}$ always points towards the outside of S.

Let $\mathbf{F}(x, y, z)$ be a continuous vector field in space, and S an oriented surface. We define

$$\textsf{\textbf{flux of }} F \textsf{\textbf{ through }} S \ = \ \iint_S (\mathbf{F}\cdot\mathbf{n})\, dS \ = \ \iint_S \mathbf{F}\cdot d\mathbf{S} \ ; \tag{3}$$



the two integrals are the same, but the second is written using the common and suggestive abbreviation $d\mathbf{S} = \mathbf{n}\, dS$.

If $\mathbf{F}$ represents the velocity field for the flow of an incompressible fluid of density 1, then $\mathbf{F}\cdot\mathbf{n}$ represents the component of the velocity in the positive perpendicular direction to the surface, and $\mathbf{F}\cdot\mathbf{n}\, dS$ represents the flow rate across the little infinitesimal piece of surface having area dS. The integral in (3) adds up these flows across the pieces of surface, so that we may interpret (3) as saying

$$\textit{flux of } F \textit{ through } S \;=\; \textit{net flow rate across } S, \tag{4}$$

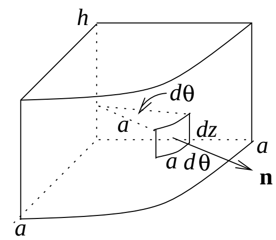
where we count flow in the direction of $\mathbf{n}$ as positive, flow in the opposite direction as negative. More generally, if the fluid has varying density, then the right side of (4) is the net mass transport rate of fluid across S (per unit area, per time unit).

If $\mathbf{F}$ is a force field, then nothing is physically flowing, and one just uses the term "flux" to denote the surface integral, as in (3).

2. Flux through a cylinder and sphere.

We now show how to calculate the flux integral, beginning with two surfaces where $\mathbf{n}$ and dS are easy to calculate — the cylinder and the sphere.

Example 1. Find the flux of $\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$ outward through the portion of the cylinder $x^2 + y^2 = a^2$ in the first octant and below the plane $z = h$.



Solution. The piece of cylinder is pictured. The word "outward" suggests that we orient the cylinder so that $\mathbf{n}$ points outward, i.e., away from the z-axis. Since by inspection $\mathbf{n}$ is radially outward and horizontal,

$$\mathbf{n} \;=\; \frac{x\,\mathbf{i} + y\,\mathbf{j}}{a}\;. \tag{5}$$

(This is the outward normal to the circle $x^2 + y^2 = a^2$ in the xy-plane; $\mathbf{n}$ has no z-component since it is horizontal. We divide by a to make its length 1.)

To get dS, the infinitesimal element of surface area, we use cylindrical coordinates to parametrize the cylinder:

$$x = a\cos\theta, \qquad y = a\sin\theta \qquad z = z\;. \tag{6}$$

As the parameters θ and z vary, the whole cylinder is traced out ; the piece we want satisfies $0 \le \theta \le \pi/2,\ 0 \le z \le h$. The natural way to subdivide the cylinder is to use little pieces of curved rectangle like the one shown, bounded by two horizontal circles and two vertical lines on the surface. Its area dS is the product of its height and width:

$$dS = dz \cdot a\,d\theta\;. \tag{7}$$

Having obtained $\mathbf{n}$ and dS, the rest of the work is routine. We express the integrand of our surface integral (3) in terms of z and θ:

$$\begin{aligned}
\mathbf{F}\cdot\mathbf{n}\,dS \;&=\; \frac{zx + xy}{a}\cdot a\,dz\,d\theta\;, && \text{by (5) and (7);}\\
&=\; (az\cos\theta + a^2\sin\theta\cos\theta)\,dz\,d\theta, && \text{using (6).}
\end{aligned}$$

This last step is essential, since the dz and $d\theta$ tell us the surface integral will be calculated in terms of z and θ, and therefore the integrand must use these variables also. We can now calculate the flux through S:

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\,dS \;&=\; \int_0^{\pi/2}\!\!\int_0^h (az\cos\theta + a^2\sin\theta\cos\theta)\,dz\,d\theta\\
\text{inner integral} \;&=\; \frac{ah^2}{2}\cos\theta + a^2h\sin\theta\cos\theta\\
\text{outer integral} \;&=\; \left[\frac{ah^2}{2}\sin\theta + a^2h\frac{\sin^2\theta}{2}\right]_0^{\pi/2} \;=\; \frac{ah}{2}(a+h)\;.
\end{aligned}$$

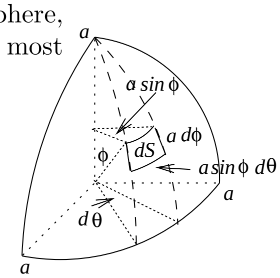
Example 2. Find the flux of $\mathbf{F} = xz\,\mathbf{i} + yz\,\mathbf{j} + z^2\,\mathbf{k}$ outward through that part of the sphere $x^2 + y^2 + z^2 = a^2$ lying in the first octant $(x, y, z, \geq 0)$.

Solution. Once again, we begin by finding $\mathbf{n}$ and dS for the sphere. We take the outside of the sphere as the positive side, so $\mathbf{n}$ points radially outward from the origin; we see by inspection therefore that

$$\mathbf{n} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a}, \tag{8}$$

where we have divided by a to make $\mathbf{n}$ a unit vector.

To do the integration, we use spherical coordinates ρ, ϕ, θ. On the surface of the sphere, $\rho = a$, so the coordinates are just the two angles ϕ and θ. The area element dS is most easily found using the volume element:



$$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = dS \cdot d\rho \quad = \quad \text{area} \cdot \text{thickness}$$

so that dividing by the thickness $d\rho$ and setting $\rho = a$, we get

$$dS = a^2 \sin\phi\, d\phi\, d\theta. \tag{9}$$

Finally since the area element dS is expressed in terms of ϕ and θ, the integration will be done using these variables, which means we need to express x, y, z in terms of ϕ and θ. We use the formulas expressing Cartesian in terms of spherical coordinates (setting $\rho = a$ since (x, y, z) is on the sphere):

$$x = a\sin\phi\cos\theta, \qquad y = a\sin\phi\sin\theta, \qquad z = a\cos\phi\ . \tag{10}$$

We can now calculate the flux integral (3). By (8) and (9), the integrand is

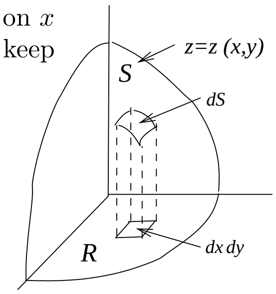
$$\mathbf{F}\cdot\mathbf{n}\,dS \;=\; \frac{1}{a}\,(x^2z + y^2z + z^2z)\cdot a^2\sin\phi\,d\phi\,d\theta\ .$$

Using (10), and noting that $x^2 + y^2 + z^2 = a^2$, the integral becomes

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\;dS \;&=\; a^4\int_0^{\pi/2}\int_0^{\pi/2}\cos\phi\sin\phi\,d\phi\,d\theta \\ &=\; a^4\,\frac{\pi}{2}\,\frac{1}{2}\,\sin^2\phi\Big]_0^{\pi/2} \;=\; \frac{\pi a^4}{4}\ .\end{aligned}$$

3. Flux through general surfaces.

For a general surface, we will use xyz-coordinates. It turns out that here it is simpler to calculate the infinitesimal vector $d\mathbf{S} = \mathbf{n}\,dS$ directly, rather than calculate $\mathbf{n}$ and dS separately and multiply them, as we did in the previous section. Below are the two standard forms for the equation of a surface, and the corresponding expressions for $d\mathbf{S}$. In the first we use z both for the dependent variable and the function which gives its dependence on x and y; you can use $f(x,y)$ for the function if you prefer, but that's one more letter to keep track of.

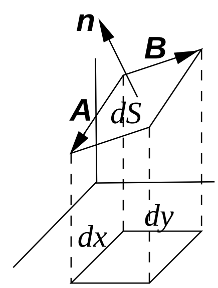


(11a) $\qquad z = z(x,y), \qquad d\mathbf{S} = (-z_x\,\mathbf{i} - z_y\,\mathbf{j} + \mathbf{k}\,)\,dx\,dy \qquad$ ($\mathbf{n}$ points "up")

(11b) $\qquad F(x,y,z) = c, \qquad d\mathbf{S} = \pm\dfrac{\nabla F}{F_z}\,dx\,dy \qquad$ (choose the right sign);

Derivation of formulas for $d\mathbf{S}$.

Refer to the pictures at the right. The surface S lies over its projection R, a region in the xy-plane. We divide up R into infinitesimal rectangles having area $dx\,dy$ and sides parallel to the xy-axes — one of these is shown. Over it lies a piece dS of the surface, which is approximately a parallelogram, since its sides are approximately parallel.

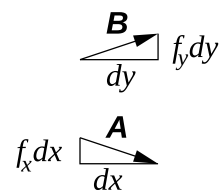


The infinitesimal vector $d\mathbf{S} = \mathbf{n}\,dS$ we are looking for has

direction: perpendicular to the surface, in the "up" direction;
magnitude: the area dS of the infinitesimal parallelogram.

This shows our infinitesimal vector is the cross-product

$$d\mathbf{S} = \mathbf{A} \times \mathbf{B}$$



where $\mathbf{A}$ and $\mathbf{B}$ are the two infinitesimal vectors forming adjacent sides of the parallelogram. To calculate these vectors, from the definition of the partial derivative, we have

$\mathbf{A}$ lies over the vector $dx\,\mathbf{i}$ and has slope f_x in the $\mathbf{i}$ direction, so $\mathbf{A} = dx\,\mathbf{i} + f_x\,dx\,\mathbf{k}$;
$\mathbf{B}$ lies over the vector $dy\,\mathbf{j}$ and has slope f_y in the $\mathbf{j}$ direction, so $\mathbf{B} = dy\,\mathbf{j} + f_y\,dy\,\mathbf{k}$.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ dx & 0 & f_x dx \\ 0 & dy & f_y dy \end{vmatrix} = (-f_x\,\mathbf{i} - f_y\,\mathbf{j} + \mathbf{k}\,)\,dx\,dy\ ,$$

which is (11a).

To get (11b) from (11a), , our surface is given by

$$(12) \qquad F(x,y,z) = c, \qquad z = z(x,y)$$

where the right-hand equation is the result of solving $F(x,y,z) = c$ for z in terms of the independent variables x and y. We differentiate the left-hand equation in (12) with respect to the independent variables x and y, using the chain rule and remembering that $z = z(x,y)$:

$$F(x,y,z) = c \quad \Rightarrow \quad F_x\frac{\partial x}{\partial x} + F_y\frac{\partial y}{\partial x} + F_z\frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad F_x + F_z\frac{\partial z}{\partial x} = 0$$

from which we get

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \qquad \text{and similarly,} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}\ .$$

Therefore by (11a),

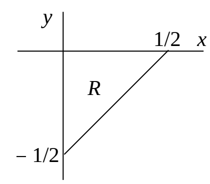
$$d\mathbf{S} \;=\; \left(-\frac{\partial z}{\partial x}\,\mathbf{i} - \frac{\partial z}{\partial y}\,\mathbf{j} + 1\right) dx\,dy \;=\; \left(\frac{F_x}{F_z}\,\mathbf{i} + \frac{F_y}{F_z}\,\mathbf{j} + 1\right) dx\,dy \;=\; \frac{\nabla F}{F_z}\,dx\,dy\ ,$$

which is (11b).

Example 3. The portion of the plane $2x - 2y + z = 1$ lying in the first octant forms a triangle S. Find the flux of $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ through S; take the positive side of S as the one where the normal points "up".

Solution. Writing the plane in the form $z = 1 - 2x + 2y$, we get using (11a),

$$\begin{aligned} d\mathbf{S} \;&=\; (2\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k})\,dx\,dy\ , \qquad \text{so} \\ \iint_S \mathbf{F}\cdot d\mathbf{S} \;&=\; \iint_S (2x - 2y + z)\,dy\,dx \\ &=\; \iint_R \big(2x - 2y + (1 - 2x + 2y)\big)\,dy\,dx\ , \end{aligned}$$



where R is the region in the xy-plane over which S lies. (Note that since the integration is to be in terms of x and y, we had to express z in terms of x and y for this last step.) To see what R is explicitly, the plane intersects the three coordinate axes respectively at $x = 1/2,\ y = -1/2,\ z = 1$. So R is the region pictured; our integral has integrand 1, so its value is the area of R, which is $1/8$.

Remark. When we write $z = f(x, y)$ or $z = z(x, y)$, we are agreeing to parametrize our surface using x and y as parameters. Thus the flux integral will be reduced to a double integral over a region R in the xy-plane, involving only x and y. Therefore you must *get rid of z by using the relation $z = z(x, y)$* after you have calculated the flux integral using (11a). Then determine the region R (the projection of S onto the xy-plane), and supply the limits for the iterated integral over R.

Example 4. Set up a double integral in the xy-plane which gives the flux of the field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ through that portion of the ellipsoid $4x^2 + y^2 + 4z^2 = 4$ lying in the first octant; take $\mathbf{n}$ in the "up" direction.

Solution. Using (11b), we have $d\,\mathbf{S} = \dfrac{\langle 8x, 2y, 8z\rangle}{8z}\,dx\,dy$. Therefore

$$\iint_S \mathbf{F}\cdot d\,\mathbf{S} = \iint_S \frac{8x^2 + 2y^2 + 8z^2}{8z}\,dx\,dy = \iint_S \frac{1}{z}\,dx\,dy = \iint_R \frac{dx\,dy}{\sqrt{1 - x^2 - (y/2)^2}}\ ,$$

where R is the portion of the ellipse $4x^2 + y^2 = 4$ lying the the first quadrant.

The double integral would be most simply evaluated by making the change of variable $u = y/2$, which would convert it to a double integral over a quarter circle in the xu-plane easily evaluated by a change to polar coordinates.

3. General surface integrals.*

The surface integral $\iint_S f(x,y,z)\,dS$ that we introduced at the beginning can be used to calculate things other than flux.

a) **Surface area.** We let the function $f(x,y,z)=1$. Then the area of $S\ =\ \iint_S dS$.

b) **Mass, moments, charge.** If S is a thin shell of material, of uniform thickness, and with density (in gms/unit area) given by $\delta(x,y,z)$, then

$$\text{mass of } S\ =\ \iint_S \delta(x,y,z)\,dS, \tag{13}$$

$$x\text{-component of center of mass}\ =\ \overline{x}\ =\ \frac{1}{\text{mass } S}\iint_S x\cdot\delta\,dS \tag{14}$$

with the y- and z-components of the center of mass defined similarly. If $\delta(x,y,z)$ represents an electric charge density, then the surface integral (13) will give the total charge on S.

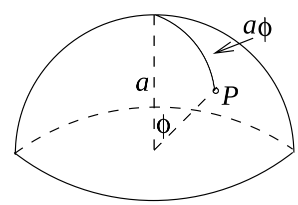
c) **Average value.** The average value of a function $f(x,y,z)$ over the surface S can be calculated by a surface integral:

$$\text{average value of } f \text{ on } S\ =\ \frac{1}{\text{area } S}\iint_S f(x,y,z)\,dS\ . \tag{15}$$

Calculating general surface integrals; finding dS.

To evaluate general surface integrals we need to know dS for the surface. For a sphere or cylinder, we can use the methods in section 2 of this chapter.

Example 5. Find the average distance along the earth of the points in the northern hemisphere from the North Pole. (Assume the earth is a sphere of radius a.)



Solution. — We use (15) and spherical coordinates, choosing the coordinates so the North Pole is at $z=a$ on the z-axis. The distance of the point (a,ϕ,θ) from $(a,0,0)$ is $a\phi$, measured along the great circle, i.e., the longitude line — see the picture). We want to find the average of this function over the upper hemisphere S. Integrating, and using (9), we get

$$\iint_S a\phi\,dS\ =\ \int_0^{2\pi}\int_0^{\pi/2} a\phi a^2\sin\phi\,d\phi\,d\theta\ =\ 2\pi a^3\int_0^{\pi/2}\phi\,\sin\phi\,d\phi\ =\ 2\pi a^3\ .$$

(The last integral used integration by parts.) Since the area of $S=2\pi a^2$, we get using (15) the striking answer: average distance $=a$.

For more general surfaces given in xyz-coordinates, since $d\mathbf{S}=\mathbf{n}\,dS$, the area element dS is the magnitude of $d\mathbf{S}$. Using (11a) and (11b), this tells us

$$z=z(x,y), \qquad dS=\sqrt{z_x^2+z_y^2+1}\,dx\,dy \tag{16a}$$

$$F(x,y,z)=c, \qquad dS=\frac{|\nabla F|}{|F_z|}\,dx\,dy \tag{16b}$$

Example 6. The area of the piece S of $z=xy$ lying over the unit circle R in the xy-plane is calculated by (a) above and (16a) to be:

$$\iint_S dS=\iint_R\sqrt{y^2+x^2+1}\,dx\,dy=\int_0^{2\pi}\int_0^1\sqrt{r^2+1}\,r\,dr\,d\theta=2\pi\cdot\frac{1}{3}(r^2+1)^{3/2}\Big]_0^1=\frac{2\pi}{3}(2\sqrt{2}-1).$$

Exercises: Section 6B