

V8. Vector Fields in Space

Just as in Section V1 we considered vector fields in the plane, so now we consider vector fields in three-space. These are fields given by a vector function of the type

$$(1) \quad \mathbf{F}(x, y, z) = M(x, y, z) \mathbf{i} + N(x, y, z) \mathbf{j} + P(x, y, z) \mathbf{k} .$$

Such a function assigns the vector $\mathbf{F}(x_0, y_0, z_0)$ to a point (x_0, y_0, z_0) where $M, N,$ and P are all defined. We place the vector so its tail is at (x_0, y_0, z_0) , and in this way get the vector field. Such a field in space looks a little like the interior of a haystack.

As before, we say \mathbf{F} is *continuous* in some domain D of 3-space (we will usually use “domain” rather than “region”, when referring to a portion of 3-space) if $M, N,$ and P are continuous in that domain. We say \mathbf{F} is *continuously differentiable* in the domain D if all nine first partial derivatives

$$M_x, M_y, M_z; N_x, N_y, N_z; P_x, P_y, P_z$$

exist and are continuous in D .

Again as before, we give two physical interpretations for such a vector field.

The three-dimensional **force fields** of different sorts — gravitational, electrostatic, electromagnetic — all give rise to such a vector field: at the point (x_0, y_0, z_0) we place the vector having the direction and magnitude of the force which the field would exert on a unit test particle placed at the point.

The three-dimensional **flow fields** and **velocity fields** arising from the motion of a fluid in space are the other standard example. We assume the motion is steady-state (i.e., the direction and magnitude of the flow at any point does not change over time). We will call this a *three-dimensional flow*.

As before, we allow sources and sinks — places where fluid is being added to or removed from the flow. Obviously, we can no longer appeal to people standing overhead pouring fluid in at various points (they would have to be aliens in four-space), but we could think of thin pipes inserted into the domain at various points adding or removing fluid.

The velocity field of such a flow is defined just as it was previously: $\mathbf{v}(x, y, z)$ gives the direction and magnitude (speed) of the flow at (x, y, z) .

The flow field $\mathbf{F} = \delta \mathbf{v}$, where $\delta(x, y, z)$ is the density) may be similarly interpreted:

$$(2) \quad \begin{aligned} \text{dir } \mathbf{F} &= \text{the direction of flow} \\ |\mathbf{F}| &= \text{mass transport rate (per unit area) at } (x, y, z) \text{ in the flow direction;} \end{aligned}$$

that is, $|\mathbf{F}|$ is the rate per unit area at which mass is transported across a small piece of plane perpendicular to the flow at the point (x, y, z) .

The derivation of this interpretation is exactly as in Sections V1 and V3, replacing the small line segment Δl by a small plane area ΔA perpendicular to the flow.

Example 1. Find the three-dimensional electrostatic force field \mathbf{F} arising from a unit positive charge placed at the origin, given that in suitable units \mathbf{F} is directed radially outward from the origin and has magnitude $1/\rho^2$, where ρ is the distance from the origin.

Solution. The vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ with tail at (x, y, z) is directed radially outward and has magnitude ρ . Therefore

$$\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\rho^3}, \quad \rho = \sqrt{x^2 + y^2 + z^2}$$

Example 2. a) Find the velocity field of a fluid rotating with constant angular velocity ω around the z -axis, in the direction given by the right-hand rule (right-hand fingers curl in direction of flow when thumb points in the \mathbf{k} -direction).

b) Find the analogous field if the flow is rotating about the y -axis.

Solution. a) The flow doesn't depend on z — it is really just a two-dimensional problem, whose solution is the same as before (section V1, Example 4):

$$\mathbf{F}(x, y, z) = \omega(-y\mathbf{i} + x\mathbf{j}).$$

b) If the axis of flow is the y -axis, the flow will have no \mathbf{j} -component and will not depend on y . However, by the right-hand rule, the flow in the xz -plane is *clockwise*, when the positive x and z axes are drawn so as to give a right-handed system. Thus

$$\mathbf{F}(x, y, z) = \omega(z\mathbf{i} - x\mathbf{k}).$$

Example 3. Find the three-dimensional flow field of a gas streaming radially outward with constant velocity from a source at the origin of constant strength.

Solution. This is like the corresponding two-dimensional problem (section V1, Example 3), except that the area of a sphere increases like the *square* of its radius. Therefore, to maintain constant velocity, the density of flow must decrease like $1/\rho^2$ as you go out from the origin; letting δ be the density and c_i be constants, we get

$$\mathbf{F}(x, y, z) = \delta\mathbf{v} = \frac{c_1}{\rho^2} \frac{c_2(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{\rho} = \frac{c(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{\rho^3}.$$

Notice that in the three-dimensional case, this field is the same as the one in Example 1 above, with the magnitude falling off like $1/\rho^2$. For the two-dimensional case, the analogue of a point fluid source at the origin is not a point charge at the origin, but a uniform charge along a vertical wire; both give the field whose magnitude falls off like $1/r$.

Exercises: Section 6A