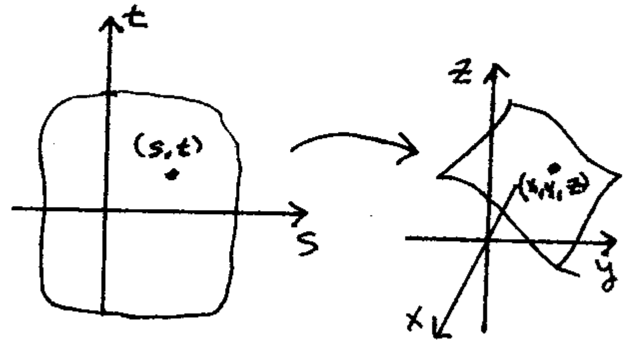
Math 18.02 – Notes on parameterized surfaces, tangent vectors, and partial derivatives

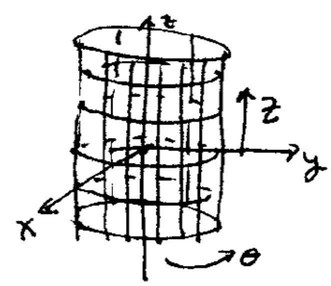
Parameterized surfaces and tangent vectors

Just as we can parameterize a curve (one degree of freedom) using a single parameter as $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, we can do a similar construction for a surface (two degrees of freedom). The basic idea is to come up with a one-to-one correspondence between locations in a two-dimensional parameter space and the given surface. As with curves, there are many ways to do this for any given surface.



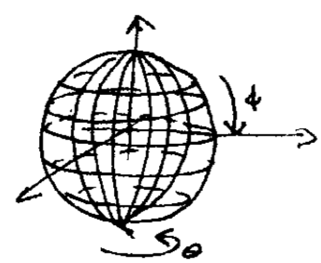
The basic idea is that we will express $\mathbf{r}(s,t) = \langle x(s,t), y(s,t), z(s,t) \rangle$ using two independent parameters. Varying each parameter independently will allow the freedom to move about on the given surface. Varying just one parameter at a time will produce parameterized curves embedded in the surface. We'll discuss surface in greater detail later in the course, but for now consider the following three examples:

Cylinder of radius *R*: Using polar coordinates to describe a circle for x and y and allowing the z coordinate to vary freely, we can describe the cylinder with equation $x^2 + y^2 = R^2$ parametrically using θ and z as independent parameters:



$\left\{ \begin{array}{l} x = R\cos\theta \\ y = R\sin\theta \\ z = z \end{array} \right\}$. Expressed in this way, z does double-duty as both a coordinate as a parameter. While this may make intuitive sense, it's generally best to distinguish the parameters as things you choose and the coordinates that then correspond to a given choice of parameters. We might therefore express the parametrization as $\mathbf{r}(\theta,t) = \langle R\cos\theta, R\sin\theta, t \rangle$, but often we'll simply write $\mathbf{r}(\theta,z) = \langle R\cos\theta, R\sin\theta, z \rangle$. Note that if we fix a value of z and vary θ only, we produce circular curves at the given z level. Similarly, if we fix θ and vary z only we'll get the vertical lines that are also embedded in the cylinder.

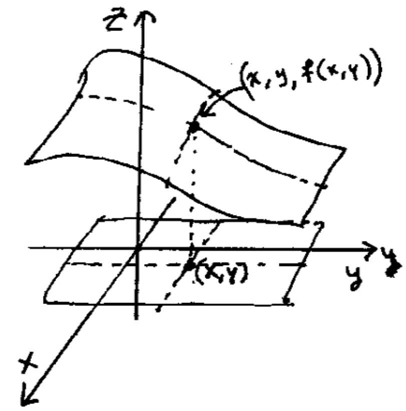
Sphere of radius *R*: If you recall the discussion of spherical coordinates from Lecture #1, we had the relations $\left\{ \begin{array}{l} x = \rho\cos\theta\sin\phi \\ y = \rho\sin\theta\sin\phi \\ z = \rho\cos\phi \end{array} \right\}$. Fixing the radius at $\rho = R$, we get a sphere described parametrically as $\left\{ \begin{array}{l} x = R\cos\theta\sin\phi \\ y = R\sin\theta\sin\phi \\ z = R\cos\phi \end{array} \right\}$. We can express this as



$\mathbf{r}(\phi,\theta) = \langle R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi \rangle$. We get the entire sphere by varying $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$. The one-to-one correspondence breaks down at the poles, but otherwise it's a one-to-one correspondence between pairs of parameters and points on the sphere. Note that if we fix the azimuth angle θ (longitude) and choose to think of the parameter ϕ as "time", then by varying this parameter we'll be heading south along a longitude. By treating θ as constant, we can differentiate to get the velocity vector $\mathbf{v}_\phi = \langle R\cos\theta\cos\phi, R\sin\theta\cos\phi, -R\sin\phi \rangle$ which basically says which way is "south". Similarly we could have fixed the latitude by fixing $\phi = \text{constant}$ and thinking of the parameter θ as "time". Differentiating with respect to this parameter would then yield the "east" velocity vector $\mathbf{v}_\theta = \langle -R\sin\theta\sin\phi, R\cos\theta\sin\phi, 0 \rangle$. Note that the *z*-component would be zero which makes sense if you're heading east along a latitude.

Graph of a function $z = f(x,y)$

Given a reasonably well-behaved function $f(x,y)$, its graph will generally yield a surface with a natural one-to-one correspondence between any (x,y) and the point $(x,y,f(x,y))$ on its graph.



As parametric equations we might write: $\left\{\begin{matrix} x = x \\ y = y \\ z = f(x,y) \end{matrix}\right\}$.

As a vector-valued function we would write $\mathbf{r}(x,y) = \langle x, y, f(x,y) \rangle$.

Again, we probably should express this as $\mathbf{r}(s,t) = \langle s, t, f(s,t) \rangle$ to keep clear the distinction between parameters and coordinates, but as long as you're clear about the context this isn't really necessary.

Looking at the sketch of the graph, it should be clear that each of the cross-section curves pictured will have well-defined slopes just as long as the graph surface is smooth. We'll formalize that in the next lecture when we discuss in greater the ideas of limits, continuity, and differentiability, but for now we can simply note that these slopes can be calculated by simply varying just one of the independent variables (parameters) at a time.

Partial derivatives

Referring to the previous example, we define the partial derivatives of $f(x,y)$ as follows:

$\dfrac{\partial f}{\partial x} = \lim\limits_{\Delta x \to 0} \left(\dfrac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \right) = f_x(x,y)$ is the **partial derivative of *f* with respect to *x***

$\dfrac{\partial f}{\partial y} = \lim\limits_{\Delta x \to 0} \left(\dfrac{f(x, y+\Delta y) - f(x,y)}{\Delta y} \right) = f_y(x,y)$ is the **partial derivative of *f* with respect to *y***

These represent, respectively, the slopes of tangent lines to the cross-section curves in the graph of $f(x,y)$ as we vary just x and just y holding the other variable fixed. The same idea extends to functions of any number of variables. To calculate the partial derivative with respect to one independent variable, you treat any other variables as though they were constant. In Latin, the expression is *ceteris paribus* meaning "with other things the same" or "all other things being equal".

Generally we will rarely need to use this formal definition to calculate partial derivatives. By simply understanding what the definitions are really saying, we can just use familiar rules for differentiation by literally "treating the other variables as though constant".

For example, if $f(x,y) = x^2 y + 7y^3$ we easily calculate that $\dfrac{\partial f}{\partial x} = 2xy$ and $\dfrac{\partial f}{\partial y} = x^2 + 21y^2$.

Finally, we can take a vector approach with the parameterization $\mathbf{r}(x,y) = \langle x, y, f(x,y) \rangle$ to produce

$\dfrac{\partial \mathbf{r}}{\partial x} = \langle 1, 0, f_x \rangle$ as a tangent vector to the cross-section where only x is varied;

$\dfrac{\partial \mathbf{r}}{\partial y} = \langle 0, 1, f_y \rangle$ as a tangent vector to the cross-section where only y is varied.

The cross product of these two tangent vectors to the graph surface will then give a normal vector to the graph at any point on the graph: $\dfrac{\partial \mathbf{r}}{\partial x} \times \dfrac{\partial \mathbf{r}}{\partial y} = \langle 1, 0, f_x(x_0,y_0) \rangle \times \langle 0, 1, f_y(x_0,y_0) \rangle = \langle -f_x(x_0,y_0), -f_y(x_0,y_0), 1 \rangle = \mathbf{n}$.

We can then use this normal vector and the point $(x_0, y_0, f(x_0, y_0))$ to get an equation for a tangent plane to the graph: $\langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1 \rangle \cdot \langle x - x_0, y - y_0, z - f(x_0, y_0) \rangle = 0$ or, after solving for z:

$$\boxed{z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$ (equation for tangent plane to graph)

Once we clear up a few ideas about what it means to be differentiable, we will then be able to say that for any (x, y) near (x_0, y_0) it should be the tangent plane provides a very good linear approximation to the actual graph.

That is:

$$\boxed{f(x, y) \cong f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$ (linear approximation for (x, y) near (x_0, y_0))

Notes by Robert Winters