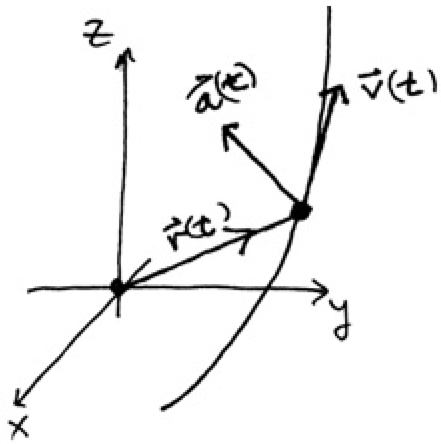
Parameterized Curves and Vector-Valued Functions

Vector-valued functions abound in physics and in the study of the geometry of curves and surfaces. This includes such physically important examples as **force**, the **angular momentum** vector, **torque**, and the most basic notions of **position**, **velocity**, and **acceleration**.

If, for the moment, we focus primarily on parameterized curves (or paths) where the parameter is the time t, we can describe motion (in $\mathbf{R}^3$, but the ideas are the same in $\mathbf{R}^2$) by specifying the position at any time t by its coordinates $(x(t), y(t), z(t))$. We can then define the **position vector** (relative to the origin) as the vector $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ from the origin to this point. The tip of the position vector will then trace out the path.

We can speak of the derivative of this or any other vector-valued function by simply taking the derivatives of the respective component functions. A simple geometric (and calculus) argument shows that the derivative $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle = \mathbf{v}(t)$ is necessarily tangent to the path in the direction of motion (or it's the zero vector) and its units are in terms of distance per time. We define $\mathbf{v}(t) = \mathbf{r}'(t)$ to be the **velocity vector** for this parameterized curve. Similarly, we can define the **acceleration vector** $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$ as the derivative of the velocity vector. The acceleration is generally in the direction of the turn unless the path is a straight line.



We also define the **speed** as the magnitude of the velocity vector, i.e. $\|\mathbf{v}(t)\|$. If we think of a parameterized curve as define starting with time $t = a$, we can let s denote the arclength of the curve to any other point on the path. We can then, alternatively, also think of the speed simply as $\frac{ds}{dt}$, the time rate of change of distance traveled, and we can conclude that $\|\mathbf{v}(t)\| = \frac{ds}{dt}$. In terms of differentials, we can then write $ds = \|\mathbf{v}(t)\|\,dt$ and, using a little bit of basic integral calculus, conclude that the **total arclength** traveled along a curve for $a \le t \le b$ must be $L = \int_a^b \|\mathbf{v}(t)\|\,dt$.

It's worth noting that since $\|\mathbf{v}(t)\|$ almost always involves square roots, <u>this arclength integral tends to be one of the more difficult integrals to calculate</u> using integration techniques and it is often best to use <u>numerical integration</u> in its calculation.

Whether we're considering a parameterized curve or any other vector-valued function, it's useful to know some basic rules of differentiation, especially when working with sums, dot products, cross products, and compositions.

Basic Rules of Differentiation for Vector-Valued Functions:

If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are vector-valued functions, $f(t)$ is any scalar-valued function, and c is any constant, then:

1) $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$ [The derivative of a sum is the sum of the derivatives.]
2) $\frac{d}{dt}[c\,\mathbf{u}(t)] = c\,\mathbf{u}'(t)$ [Scalars pass through derivatives just as with ordinary differentiation.]
3) $\frac{d}{dt}[f(t)\,\mathbf{u}(t)] = f(t)\,\mathbf{u}'(t) + f'(t)\cdot\mathbf{u}(t)$ [Product Rule for scalar multiplication.]
4) $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}(t)\cdot\mathbf{v}'(t) + \mathbf{u}'(t)\cdot\mathbf{v}(t)$ [Product Rule for dot products.]
5) $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}(t)\times\mathbf{v}'(t) + \mathbf{u}'(t)\times\mathbf{v}(t)$ [Product Rule for cross products.]
6) $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\,\mathbf{u}'(f(t))$ [Chain Rule for compositions.]

Each of these follows by applying the familiar rules of differentiation in the components. In particular, note that since the cross product is not commutative (it's anti-commutative), the order of the factors matters.

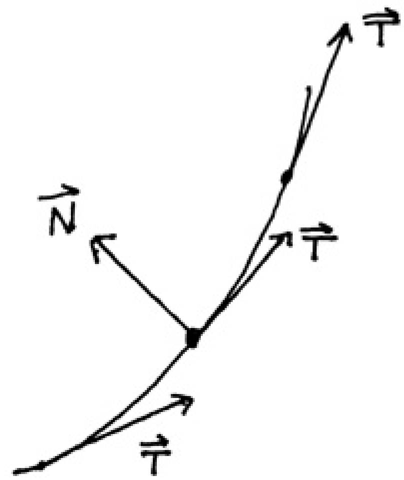
Velocity, unit tangent vector, unit normal vector, and curvature

For a parameterized curve described by $\mathbf{r}(t)$, as long as its velocity vector $\mathbf{v}(t)$ is nonzero, this velocity vector will be tangent to the path for all *t* in the direction of the path. We can divide out its magnitude (the speed) to define the **unit tangent vector** $\mathbf{T}(t)=\dfrac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}$. <u>This is a unit vector in the direction of the path</u>.

[It's worth noting that we can use the position vector at a given time *t* to determine a point on the path and either the velocity vector or the unit tangent vector at that time *t* to determine a tangent vector to the path at the given point. These can ten be used to write down a parameterization of the tangent line to the path at that point.]

We can use the Differentiation Rules to draw some addition conclusions and make a few new definitions.

Since **T** is a unit vector, it follows that $\mathbf{T}\cdot\mathbf{T}=\|\mathbf{T}\|^2=1$. We can then differentiate both sides (with respect to *t*) to get that $\mathbf{T}\cdot\mathbf{T}'+\mathbf{T}'\cdot\mathbf{T}=2\mathbf{T}\cdot\mathbf{T}'=0$. So $\mathbf{T}\cdot\mathbf{T}'=0$. This means that either $\mathbf{T}'=\mathbf{0}$ (which would be the case for a straight line path) or, more generally, $\mathbf{T}'\perp\mathbf{T}$. That is, the rate of change of the (unit) direction vector is perpendicular to the path for all *t*. [<u>Note</u>: We would draw the same conclusion regardless of the parameter – a useful thing in the case where we may choose to parameterize the curve by arclength *s* rather than by time *t*.]



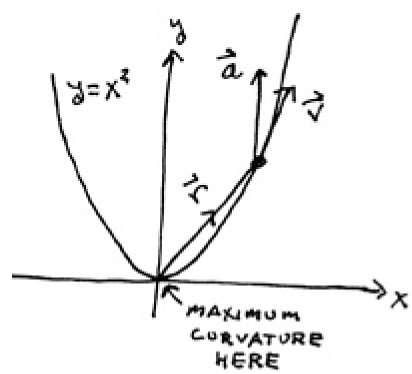
If the path is parameterized by (time) *t*, we can use the previous observation to define a **unit normal vector** to the path by $\mathbf{N}(t)=\dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$, a unit vector perpendicular to the path that points in the direction of the turn.

It is often more convenient to think of a path being <u>parameterized by the arclength</u> traveled along the path from the start (assuming, of course, that you always move in the same direction along the path and don't retrace the path). In this case we can think of the arclength *s* as a function of time *t*, i.e. $s=s(t)$. This is especially useful when thinking about the intrinsic shape of the path independent of how quickly you move along it. It then becomes natural to define the curvature in terms of the rate of change of direction with respect to distance traveled. We thus define the **curvature** κ by $\kappa=\left\|\frac{d\mathbf{T}}{ds}\right\|$. It is rarely easy to actually re-parameterize a path by its arclength, so it's helpful to be able to express the curvature (and other useful things) in terms of the original parameter *t*. To make the connection, we use the Chain Rule.

$\dfrac{d\mathbf{T}}{dt}=\dfrac{d}{dt}\big[\mathbf{T}(s(t))\big]=s'(t)\mathbf{T}'(s(t))=\dfrac{d\mathbf{T}}{ds}\dfrac{ds}{dt}=\|\mathbf{v}(t)\|\dfrac{d\mathbf{T}}{ds}$ where we've conveniently used the fact that $\|\mathbf{v}(t)\|=\frac{ds}{dt}$.

So $\dfrac{d\mathbf{T}}{ds}=\dfrac{\mathbf{T}'(t)}{\|\mathbf{v}(t)\|}$ and, therefore, the curvature may also be written as $\kappa=\left\|\dfrac{d\mathbf{T}}{ds}\right\|=\dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{v}(t)\|}$.

Example: Let's look at the familiar parabola $y=x^2$ in the *xy*-plane. This is a static, algebraic equation for the parabola, but we can also describe it parametrically by letting $x=t$ (and therefore $y=t^2$). We thus get the position vector and parameterization $\mathbf{r}(t)=\langle t,t^2\rangle$. We can then easily calculate that $\mathbf{v}(t)=\langle 1,2t\rangle$ and $\mathbf{a}(t)=\langle 0,2\rangle$. These are all shown in the accompanying sketch. We can also quickly calculate the speed $\|\mathbf{v}(t)\|=\sqrt{1+4t^2}$ as well as the unit tangent vector $\mathbf{T}(t)=\dfrac{\mathbf{v}}{\|\mathbf{v}\|}=\dfrac{\langle 1,2t\rangle}{\sqrt{1+4t^2}}=\left\langle \dfrac{1}{\sqrt{1+4t^2}},\dfrac{2t}{\sqrt{1+4t^2}}\right\rangle$. If we exercise our differentiation skills (especially the quotient rule), we can then calculate $\mathbf{T}'(t) = \left\langle \frac{-4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}} \right\rangle$. Its magnitude is then $\|\mathbf{T}'(t)\| = \frac{\sqrt{16t^2+4}}{(1+4t^2)^{3/2}} = \frac{2\sqrt{4t^2+1}}{(1+4t^2)^{3/2}} = \frac{2}{1+4t^2}$. The curvature is therefore given by $\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{v}(t)\|} = \frac{2}{(1+4t^2)^{3/2}}$.



Note that $t = 0$ corresponds to the origin $(0,0)$ and that at this point the curvature is $\kappa = 2$. Further note that for $t \neq 0$, the curvature decreases. This is consistent with the shape of the parabola.