Math 18.02 – Notes on Dot Product, Cross Product, Planes, Area, and Volumes

This lecture focuses primarily on the dot product and its many applications, especially in the measurement of angles and scalar projection and determining the equation of a plane. We also introduce the cross product in $\mathbf{R}^3$ which can be used to find a vector orthogonal (perpendicular) to any pair of nonzero, nonparallel vectors in $\mathbf{R}^3$ and can also be used in the measurement of area.

The Dot Product (or Scalar Product)

In addition to the most basic operations of scaling and vector addition (both done component-wise), the measurement of lengths and angles are facilitated by the dot product of vectors (also known as the inner product or the scalar product). The dot product can be defined in $\mathbf{R}^n$ for any n which will allow for the definition of orthogonality in any dimension.

Definition: The **dot product** of two vectors **u**, **v** in $\mathbf{R}^n$ is a scalar defined as follows:

$$\mathbf{u}\cdot\mathbf{v}=\langle u_1,u_2,\cdots,u_n\rangle\cdot\langle v_1,v_2,\cdots,v_n\rangle=u_1v_1+u_2v_2+\cdots+u_nv_n$$

Measuring the length of a vector: Note that for any vector **u** in $\mathbf{R}^n$,

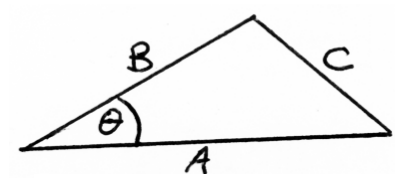
$\mathbf{u}\cdot\mathbf{u}=\langle u_1,u_2,\cdots,u_n\rangle\cdot\langle u_1,u_2,\cdots,u_n\rangle=u_1^2+u_2^2+\cdots+u_n^2=\|\mathbf{u}\|^2$, so $\boxed{\mathbf{u}\cdot\mathbf{u}=\|\mathbf{u}\|^2}$ or $\boxed{\|\mathbf{u}\|=\sqrt{\mathbf{u}\cdot\mathbf{u}}}$.

There are some easy-to-verify algebraic properties of the dot product that follow from its definition:

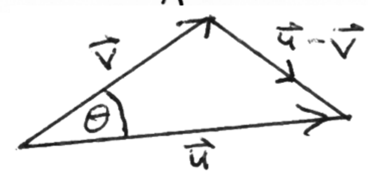
Algebraic Properties of the Dot Product: Suppose **u**, **v**, and **w** are vectors in $\mathbf{R}^n$ and that t is any scalar.

1) $\mathbf{v}\cdot\mathbf{u}=\mathbf{u}\cdot\mathbf{v}$ (symmetry, dot product is commutative)
2) $\left\{\begin{matrix}\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}\\(\mathbf{u}+\mathbf{v})\cdot\mathbf{w}=\mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}\end{matrix}\right\}$ (left and right distributive laws)
3) $(t\,\mathbf{u})\cdot\mathbf{v}=t(\mathbf{u}\cdot\mathbf{v})=\mathbf{u}\cdot(t\,\mathbf{v})$ (how the dot product behaves relative to scaling of vectors)
4) $\mathbf{u}\cdot\mathbf{u}=\|\mathbf{u}\|^2\geq 0$ for all **u** (and $\mathbf{u}\cdot\mathbf{u}=\|\mathbf{u}\|^2=0$ only for $\mathbf{u}=\mathbf{0}$)

A corollary of the Pythagorean Theorem is the **Law of Cosines**. Referring to the figure, the Law of Cosines states that $C^2=A^2+B^2-2AB\cos\theta$ where A and B are the lengths of the sides adjacent to the angle θ and C is the length of the side opposite this angle.



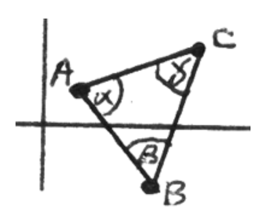
We can state a vector version of this using a modified figure. In this figure the Law of Cosines can be expressed as $\|\mathbf{u}-\mathbf{v}\|^2=\|\mathbf{u}\|^2+\|\mathbf{v}\|^2-2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$. Using the facts above, the left-hand-side gives



$\|\mathbf{u}-\mathbf{v}\|^2=(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})=\mathbf{u}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v}=\|\mathbf{u}\|^2+\|\mathbf{v}\|^2-2\mathbf{u}\cdot\mathbf{v}$. Therefore $\|\mathbf{u}\|^2+\|\mathbf{v}\|^2-2\mathbf{u}\cdot\mathbf{v}=\|\mathbf{u}\|^2+\|\mathbf{v}\|^2-2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$, and cancellation gives that $\boxed{\mathbf{u}\cdot\mathbf{v}=\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta}$. This restatement of the Law of Cosines is also known as the **geometric definition of the dot product**.

The great importance of this relation is that it connects the algebraically-defined dot product (sum of the products of the respective components) to the geometric **measurements of lengths and angles**.

Example: A triangle in $\mathbf{R}^2$ has vertices at the points with coordinates $A(1,1)$, $B(3,-2)$, and $C(4,2)$. Find the angles α,β,γ shown.



Solution: The dot product requires two vectors, so in order to proceed we have to choose vectors appropriate for each of the given angles . For example, to determine the angle α we will want to find the vectors $\mathbf{u} = \overrightarrow{AB} = \langle 2, -3 \rangle$ and $\mathbf{v} = \overrightarrow{AC} = \langle 3, 1 \rangle$ emanating <u>out</u> from this common vertex. The relation $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \alpha$ can be expressed as $\cos \alpha = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$, so $\cos \alpha = \dfrac{\langle 2, -3 \rangle \cdot \langle 3, 1 \rangle}{\sqrt{13}\sqrt{10}} = \dfrac{3}{\sqrt{130}}$ and $\boxed{\alpha = \cos^{-1}\left(\dfrac{3}{\sqrt{130}}\right) \cong 74.74°}$. Similarly, we can use the vectors $\overrightarrow{BA} = \langle -2, 3 \rangle$ and $\overrightarrow{BC} = \langle 1, 4 \rangle$ to calculate that $\cos \beta = \dfrac{10}{\sqrt{221}}$ and $\boxed{\beta = \cos^{-1}\left(\dfrac{10}{\sqrt{221}}\right) \cong 47.73°}$; and we can use the vectors $\overrightarrow{CB} = \langle -1, -4 \rangle$ and $\overrightarrow{CA} = \langle -3, -1 \rangle$ to calculate that $\cos \gamma = \dfrac{7}{\sqrt{170}}$ and $\boxed{\beta = \cos^{-1}\left(\dfrac{7}{\sqrt{170}}\right) \cong 57.53°}$. Note that $\alpha + \beta + \gamma = 180°$, as expected.

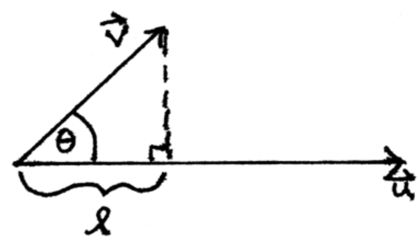
Acute, obtuse, right angles

We know that when an angle θ is acute, then $\cos\theta > 0$; when θ is obtuse, then $\cos\theta < 0$; and when θ is a right angle, then $\cos\theta = 0$. If we couple these observations with the relation $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta$, we get that

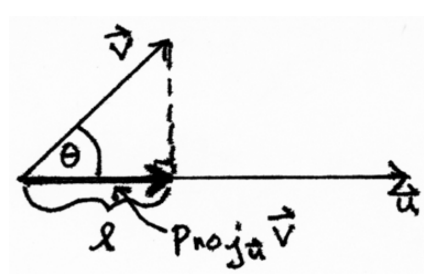
if $\mathbf{u}$, $\mathbf{v}$ in $\mathbf{R}^n$ are <u>nonzero</u> vectors emanating from a common vertex to form an angle θ, then

$\mathbf{u} \cdot \mathbf{v} > 0$ if and only if the angle θ is acute
$\mathbf{u} \cdot \mathbf{v} < 0$ if and only if the angle θ is obtuse
$\mathbf{u} \cdot \mathbf{v} = 0$ if and only if the angle θ is a right angle, i.e. $\mathbf{u} \perp \mathbf{v}$

Orthogonal Projection



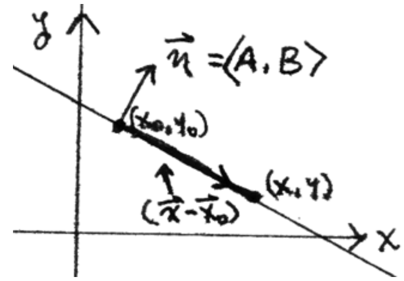
Referring to the sketch shown, the **scalar projection** of the vector **v** in the direction of the vector **u** (also called the **component** of the vector **v** in the direction of the vector **u**) is the length *l* (which will be positive for an acute angle and negative for an obtuse angle). Basic trigonometry gives that $l = \|\mathbf{v}\| \cos\theta$ and the relation $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta$ gives that $l = \|\mathbf{v}\| \cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|} = \dfrac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{u}\|} = \mathbf{v} \cdot \left(\dfrac{\mathbf{u}}{\|\mathbf{u}\|}\right)$. This latter expression enables us to express this in words simply as "**To find the component of a vector in a given direction, calculate the dot product of that vector with a <u>*unit vector*</u> in the desired direction.**" This is consistent with our previous use of the word *component*, e.g. the *x*-component of the vector $\mathbf{v} = \langle x, y, z \rangle$ is given by $\mathbf{v} \cdot \mathbf{i} = \langle x, y, z \rangle \cdot \langle 1, 0, 0 \rangle = x$. We can now, however, find the component of a vector in *any* given direction and not just in the directions of the coordinate axes.



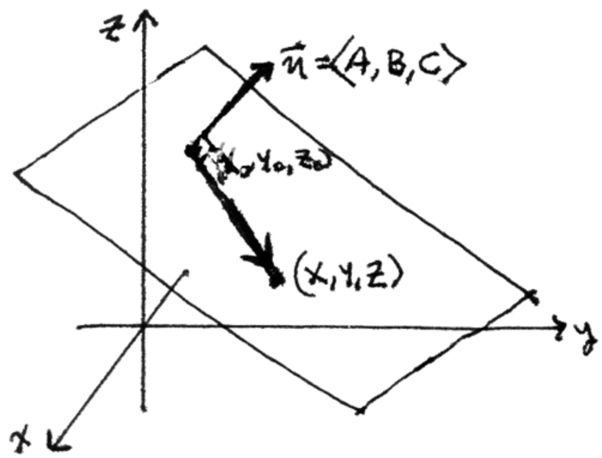
We can then use this fact to define the **vector projection** of **v** in the direction of **u** by construction it by starting with the vector u, normalizing it to get a unit vector in the same direction, and the scaling it by the value of the scalar projection to get $\boxed{\mathrm{Proj}_{\mathbf{u}}\mathbf{v} = \left(\mathbf{v} \cdot \dfrac{\mathbf{u}}{\|\mathbf{u}\|}\right)\dfrac{\mathbf{u}}{\|\mathbf{u}\|} = \left(\dfrac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{u}\|^2}\right)\mathbf{u}}$. This can be useful for expressing a vector as the sum of a "tangential component" vector and a "normal component" vector, especially in geometry and physics.

Equations for lines in $\mathbf{R}^2$ and planes in $\mathbf{R}^3$

We can use the dot product to construct an equation for a line in $\mathbf{R}^2$ given any known point (x_0, y_0) on the line and a ***normal vector*** $\mathbf{n} = \langle A, B \rangle$ perpendicular to the line. Note that for any other point (x, y) to be on this line it must be the case that the difference vector $\mathbf{x} - \mathbf{x}_0 = \langle x - x_0, y - y_0 \rangle$ is orthogonal to the normal vector $\mathbf{n} = \langle A, B \rangle$ and this will be the case if and only if $\boxed{\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0}$. If we write this out in components we get that $\langle A, B \rangle \cdot \langle x - x_0, y - y_0 \rangle = 0$ or $\boxed{A(x - x_0) + B(y - y_0) = 0}$. Note that the normal vector has slope $\frac{B}{A}$ (assuming it's neither horizontal nor vertical) and we can re-express the previous equation in the form $(y - y_0) = -\frac{A}{B}(x - x_0)$ which is just the familat point-slope form of a line if we recognize that the slope $-\frac{A}{B}$ is just the negative reciprocal of the slope $\frac{B}{A}$ of the normal vector.



The construction is essentially the same for a plane in $\mathbf{R}^3$. In this case if $\mathbf{x}_0 = \langle x_0, y_0, z_0 \rangle$ represents the position vector of any known point in the plane and if $\mathbf{n} = \langle A, B, C \rangle$ is a normal vector for the plane (note that any nonzero scalar multiple will do just as well), then if $\mathbf{x} = \langle x, y, z \rangle$ is the position vector of any other point in the plane it must necessarily be the case that the difference vector $\mathbf{x} - \mathbf{x}_0 = \langle x - x_0, y - y_0, z - z_0 \rangle$ is orthogonal to the normal vector $\mathbf{n} = \langle A, B, C \rangle$ and this will be the case if and only if $\boxed{\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0}$. If we express this in components, this becomes $\langle A, B, C \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$ or $\boxed{A(x - x_0) + B(y - y_0) + C(z - z_0) = 0}$. If we expand this out and transpose all the constants to the right-hand-side we get an equation of the form $\boxed{Ax + By + Cz = D}$ where D is a constant. Note that the components of the normal vector appear as the coefficients in this linear equation. Had we used a scalar multiple for the normal vector we would get an equivalent linear equation with these coefficients still in the same proportion as the components of the normal vector. In problems, we often jump to this form once we know the normal vector and determine D by plugging in the coordinates of the given point.



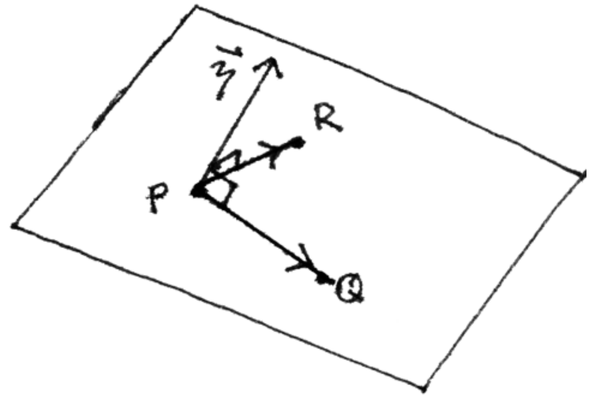
Example: Find an equation for the plane with normal vector $\mathbf{n} = \langle 2, 1, -4 \rangle$ that passes through the point with coordinates $(1, 3, 5)$.

Solution: We can simply substitute into the construction above to get that $\langle 2, 1, -4 \rangle \cdot \langle x - 1, y - 3, z - 5 \rangle = 0$ or $2(x - 1) + (y - 3) - 4(z - 5) = 0$ or $\boxed{2x + y - 4z = -15}$. Note that a plane with the equation $2x + y - 4z = 11$ would have the same (or a parallel) normal vector but shares no points with the previous plane. They are parallel planes.

The Cross Product (or Vector Product)

When find the equation of a plane we will not necessarily be provided with a known point and a convenient normal vector. We might, for example be given three points that are not co-linear, i.e. that do not all lie on a single line.

Problem: Find an equation for the plane that contains the three points $P(1, 2, -3)$, $Q(2, 1, 0)$, and $R(3, 3, 1)$.



In this case there are several productive approaches we can take. For example, we can take points pairwise to produce vectors parallel to the plane and then try to find a vector orthogonal to these two vectors. Such a vector will then serve as a normal vector for this plane. We would like to find a vector $\mathbf{n} = \langle A, B, C \rangle$ such that $\overrightarrow{PQ} \cdot \mathbf{n} = 0$ and $\overrightarrow{PR} \cdot \mathbf{n} = 0$. For the given points we have $\overrightarrow{PQ} = \langle 1, -1, 3 \rangle$ and $\overrightarrow{PR} = \langle 2, 1, 4 \rangle$, so the above conditions translate into two equations in three unknowns, namely $\left\{ \begin{matrix} A - B + 3C = 0 \\ 2A + B + 4C = 0 \end{matrix} \right\}$. This has infinitely many solutions (corresponding to that face that any scalar multiple of a normal vector will still be a normal vector), but adding them gives the relation $3A + 7C = 0$ or $3A = -7C$. If we simply choose $C = 3$, then we must necessary have $A = -7$ and we can use either of the two original equations to solve for $B = 2$. Therefore $\mathbf{n} = \langle -7, 2, 3 \rangle$ will work as a normal vector. This method gets us a solution, but it's anything but routine.

<u>Alternatively</u>, we might simply observe that the equation of the plane must be of the form $Ax + By + Cz = D$ for appropriate constants A, B, C, D. Since the three given points all presumably lie on this plane, they must all satisfy the equation for the plane, so we get three equations in the four unknowns A, B, C, D, namely $\left\{ \begin{matrix} A + 2B - 3C = D \\ 2A + B \quad\;\; = D \\ 3A + 3B + C = D \end{matrix} \right\}$ which also be written as $\left\{ \begin{matrix} A + 2B - 3C - D = 0 \\ 2A + B \quad\;\; - D = 0 \\ 3A + 3B + C - D = 0 \end{matrix} \right\}$. For those of you who are familiar with row reduction, this can be solved as $\left[\begin{array}{cccc|c} 1 & 2 & -3 & -1 & 0 \\ 2 & 1 & 0 & -1 & 0 \\ 3 & 3 & 1 & -1 & 0 \end{array} \right] \xrightarrow{RREF} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{7}{12} & 0 \\ 0 & 1 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & 0 \end{array} \right]$ to yield that $A = \frac{7}{12} D$, $B = -\frac{1}{6} D$, and $C = -\frac{1}{4} D$. If integer values are desired, we can choose $D = 12$ which then gives $A = 7$, $B = -2$, and $C = -3$. So an equation for this plane is $7x - 2y - 3z = 12$. Once again, we get a solution, but there must be a better way.

The most convenient way to do this is define the **cross product** of two vectors (defined only in $\mathbf{R}^3$). Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ in $\mathbf{R}^3$, we can use the orthogonality requirement to show that the following **cross product** will be orthogonal to both vectors:

$$\mathbf{u} \times \mathbf{v} = \langle u_1, u_2, u_3 \rangle \times \langle v_1, v_2, v_3 \rangle = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle$$

There are several different ways to express this using the definition of a 2×2 determinant, namely $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. Examining the above expression we see that:

$$\mathbf{u} \times \mathbf{v} = \left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right\rangle = \left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right\rangle$$

Note the sign switch in the middle component. This is done so that you can conveniently perform the calculation by creating a 2×3 array from the given two vectors $\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$ and then respectively covering the 1st, 2nd, and 3rd columns and calculating the determinant of the resulting 2×2 determinants (with appropriate sign switch of the middle component. For example, if $\mathbf{u} = \overrightarrow{PQ} = \langle 1, -1, 3 \rangle$ and $\mathbf{v} = \overrightarrow{PR} = \langle 2, 1, 4 \rangle$, we would get the array $\begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 4 \end{bmatrix}$ and use the procedure to calculate $\mathbf{u}\times\mathbf{v} = \langle -4-3, -(4-6), 1-(-2)\rangle = \langle -7,2,3\rangle$. This coincides with the normal vector we obtained with greater effort in our first attempt.

Some people prefer to express this procedure using $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ notation by formally calculating the 3×3 determinant $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$.

Using only this algebraic definition for the cross product, we can derive the following properties:

Algebraic Properties of the Cross Product: Suppose **u**, **v**, and **w** are vectors in $\mathbf{R}^3$ and that t is any scalar.

1) $\mathbf{v}\times\mathbf{u} = -\mathbf{u}\times\mathbf{v}$ (anticommutative) [Corollary: $\mathbf{u}\times\mathbf{u} = \mathbf{0}$ for any vector $\mathbf{u}$]
2) $\left\{\begin{matrix} \mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w} \\ (\mathbf{u}+\mathbf{v})\times\mathbf{w} = \mathbf{u}\times\mathbf{w}+\mathbf{v}\times\mathbf{w} \end{matrix}\right\}$ (left and right distributive laws)
3) $(t\,\mathbf{u})\times\mathbf{v} = t(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\times(t\,\mathbf{v})$ (how the dot product behaves relative to scaling of vectors)
4) $\mathbf{u}\times\mathbf{0} = \mathbf{0}$
5) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$ (triple scalar product)
6) $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ (triple vector product)

All of the above algebraic properties of the cross product except for the last one are straightforward. You can prove the last one by noting that the first component would be:

$$\begin{vmatrix} u_2 & u_3 \\ v_3w_1 - v_1w_3 & v_1w_2 - v_2w_1 \end{vmatrix} = u_2(v_1w_2 - v_2w_1) - u_3(v_3w_1 - v_1w_3) = u_2v_1w_2 - u_2v_2w_1 - u_3v_3w_1 + u_3v_1w_3$$
$$= u_2v_1w_2 - u_2v_2w_1 - u_3v_3w_1 + u_3v_1w_3 + u_1v_1w_1 - u_1v_1w_1 = (u_1w_1 + u_2w_2 + u_3w_3)v_1 - (u_1v_1 + u_2v_2 + u_3v_3)w_1$$
$$= (\mathbf{u}\cdot\mathbf{w})v_1 - (\mathbf{u}\cdot\mathbf{v})w_1$$

Similarly, we can show that the 2nd and 3rd components are $(\mathbf{u}\cdot\mathbf{w})v_2 - (\mathbf{u}\cdot\mathbf{v})w_2$ and $(\mathbf{u}\cdot\mathbf{w})v_3 - (\mathbf{u}\cdot\mathbf{v})w_3$.

Together these give that $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$. Physicists (and others) often refer to this property as the "BAC-CAB Rule" and express it as $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$.

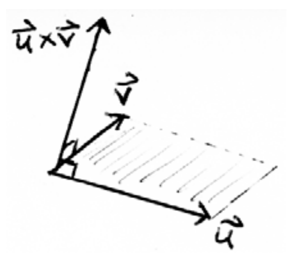
We can independent define the cross product in purely geometric terms.

Geometric definition of the cross product: Suppose **u** and **v** are vectors in $\mathbf{R}^3$. Then the cross product $\mathbf{u}\times\mathbf{v}$ is the unique vector in $\mathbf{R}^3$ such that:

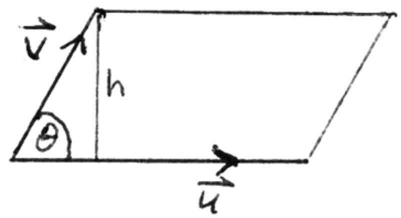
(1) $\mathbf{u}\times\mathbf{v}$ is orthogonal to both **u** and **v**;

(2) the magnitude of the cross product $\|\mathbf{u}\times\mathbf{v}\|$ is equal to the area of the parallelogram determined by **u** and **v**;

(3) $\mathbf{u}\times\mathbf{v}$ is oriented according to the Right-Hand Rule (as explained in class and elsewhere).



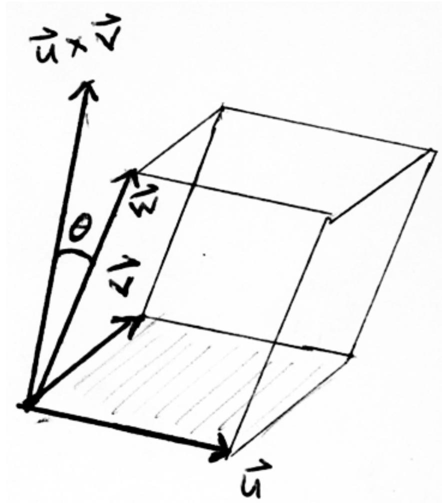
It is true that these three properties uniquely determine the cross product, and we can also easily derive the previous algebraic definition from these requirements. We can also derive these geometric properties from the algebraic definition using the previously stated algebraic properties. Specifically:



(1) $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = (\mathbf{u}\times\mathbf{u})\cdot\mathbf{v} = \mathbf{0}\cdot\mathbf{v} = 0$, so $\mathbf{u}\times\mathbf{v}$ is orthogonal to the vector **u**;
$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{v}) = \mathbf{u}\cdot\mathbf{0} = 0$, so $\mathbf{u}\times\mathbf{v}$ is orthogonal to the vector **v**.

(2) If we consider the parallelogram determined by **u** and **v** and let θ be the angle between these vectors (drawing a picture is advisable), then the area of the parallelogram will be given by
(length of base)($\perp$ height) $= \|\mathbf{u}\|\|\mathbf{v}\||\sin\theta|$. Squaring both sides gives
$(\text{Area})^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2\sin^2\theta = \|\mathbf{u}\|^2\|\mathbf{v}\|^2(1-\cos^2\theta) = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2\|\mathbf{v}\|^2\cos^2\theta = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2$.
On the other hand, $\|\mathbf{u}\times\mathbf{v}\|^2 = (\mathbf{u}\times\mathbf{v})\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\cdot[\mathbf{v}\times(\mathbf{u}\times\mathbf{v})] = \mathbf{u}\cdot[(\mathbf{v}\cdot\mathbf{v})\mathbf{u} - (\mathbf{v}\cdot\mathbf{u})\mathbf{v}] = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2$.
Therefore $(\text{Area})^2 = \|\mathbf{u}\times\mathbf{v}\|^2$, so $\text{Area} = \|\mathbf{u}\times\mathbf{v}\|$.

(3) You can easily calculate using the algebraic definition that $\mathbf{i}\times\mathbf{j} = \mathbf{k}$ which satisfies the Right-Hand Rule. Then argue using a continuity argument that if this is true for these two vectors than by continuously varying these vectors in $\mathbf{R}^3$ to align them with the given two vectors, the right-hand rule must be preserved.



Volume and the Triple Scalar Product:
Property (5) of the Algebraic Properties of the Cross Product (the triple scalar product) has an interesting geometric interpretation. Note that $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \|\mathbf{u}\times\mathbf{v}\|\|\mathbf{w}\|\cos\theta$ where θ is the angle between the vectors $\mathbf{u}\times\mathbf{v}$ and **w**. [For simplicity, we're considering the case where θ is an acute angle. If it is obtuse, everything's the same except for a change in sign.] Note that $\|\mathbf{u}\times\mathbf{v}\|$ gives the area of the parallelogram that forms the base of the ***parallelepiped*** determined by the three vectors **u**, **v**, and **w** in $\mathbf{R}^3$; and $\|\mathbf{w}\|\cos\theta$ corresponds to the perpendicular height of this parallelepiped, i.e. the scalar projection of the vector w in the direction of the vector $\mathbf{u}\times\mathbf{v}$.
Thus $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \|\mathbf{u}\times\mathbf{v}\|\left(\|\mathbf{w}\|\cos\theta\right) = (\text{area of base})(\perp\text{ height}) = \text{Volume}$ of the parallelepiped (up to sign). So the volume is $|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$ or $|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}|$.

The triple scalar product may also be calculated as $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$, a 3×3 determinant.

Notes by Robert Winters