Math 15a – Fall 2007 – Homework #8 (not to be turned in)

Section 8.1:

6. Without using technology, find an orthonormal eigenbasis for the matrix $\mathbf{A} = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$.

10. For the matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix}$, find an orthogonal matrix **S** and a diagonal matrix **D** such that $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{D}$. Do not use technology.

12. Let L from $\mathbf{R}^3$ to $\mathbf{R}^3$ be the reflection about the line spanned by $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

a. Find an orthonormal eigenbasis $\mathfrak{B}$ for L.

b. Find the matrix **B** of L with respect to $\mathfrak{B}$.

c. Find the matrix **A** of L with respect to the standard basis of $\mathbf{R}^3$.

16. a. Find the eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ with their multiplicities. Note that the algebraic multiplicity agrees with the geometric multiplicity. (Why?) *Hint*: What is the kernel of **A**?

b. Find the eigenvalues of the matrix $\mathbf{B} = \begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 3 \end{bmatrix}$ with their multiplicities. Do not use technology.

c. Use your result in part (b) to find det(**B**).

24. Consider the matrix $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Find an orthonormal eigenbasis for **A**.

36. Consider a symmetric $n \times n$ matrix **A** with $\mathbf{A}^2 = \mathbf{A}$. Is the linear transformation $T(\mathbf{x}) = \mathbf{Ax}$ necessarily the orthogonal projection onto a subspace of $\mathbf{R}^n$?

Section 8.2:

4. Determine the definiteness of the quadratic form $q(x_1, x_2) = 6x_1^2 + 4x_1x_2 + 3x_2^2$.

6. Determine the definiteness of the quadratic form $q(x_1, x_2) = 2x_1^2 + 6x_1x_2 + 4x_2^2$.

8. If **A** is a symmetric matrix, what can you say about the definiteness of $\mathbf{A}^2$? When is $\mathbf{A}^2$ positive definite?

Sketch the curves in Exercises 16 and 18. In each case, draw and label the principal axes, label the intercepts of the curve with the principal axes, and give the formula of the curve in the coordinate system defined by the principal axes.

16. $q(x_1, x_2) = x_1x_2 = 1$

18. $q(x_1, x_2) = 9x_1^2 - 4x_1x_2 + 6x_2^2 = 1$

22. On the surface $-x_1^2 + x_2^2 - x_3^2 + 10x_1x_3 = 1$, find the two points closest to the origin.

For additional practice:
Section 8.1:

3. Without using technology, find an orthonormal eigenbasis for the matrix $\mathbf{A} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$.

5. Without using technology, find an orthonormal eigenbasis for the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

15. If **A** is invertible and orthogonally diagonalizable, is $\mathbf{A}^{-1}$ orthogonally diagonalizable as well?

19. Consider a linear transformation *T* from $\mathbf{R}^m$ to $\mathbf{R}^n$. Show that there is an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ of $\mathbf{R}^m$ such that the vectors $\{L(\mathbf{v}_1), \ldots, L(\mathbf{v}_m)\}$ are orthogonal. Note that some of the vectors $L(\mathbf{v}_i)$ may be zero. *Hint*: Consider an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ for the symmetric matrix $\mathbf{A}^{\mathrm{T}}\mathbf{A}$.

29. Consider a symmetric matrix **A**. If the vector **v** is in the image of **A** and **w** is in the kernel of **A**, is **v** necessarily orthogonal to **w**? Justify your answer.

Section 8.2:

1. For the quadratic form $q(x_1, x_2) = 6x_1^2 - 7x_1x_2 + 8x_2^2$, find a symmetric matrix **A** such that $q(\mathbf{x}) = \mathbf{x} \cdot \mathbf{A}\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$.

2. For the quadratic form $q(x_1, x_2) = x_1x_2$, find a symmetric matrix **A** such that $q(\mathbf{x}) = \mathbf{x} \cdot \mathbf{A}\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$.

3. For the quadratic form $q(x_1, x_2, x_3) = 3x_1^2 + 4x_2^2 + 5x_3^2 + 6x_1x_3 + 7x_2x_3$, find a symmetric matrix **A** such that $q(\mathbf{x}) = \mathbf{x} \cdot \mathbf{A}\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$.

9. Recall that a real square matrix **A** is called *skew-symmetric* if $\mathbf{A}^{\mathrm{T}} = -\mathbf{A}$.

   a. If **A** is skew-symmetric, is $\mathbf{A}^2$ skew-symmetric as well? Or is $\mathbf{A}^2$ symmetric?

   b. If **A** is skew-symmetric, what can you say about the definiteness of $\mathbf{A}^2$? What about the eigenvalues of $\mathbf{A}^2$?

   c. What can you say about the complex eigenvalues of a skew-symmetric matrix? Which skew-symmetric matrices are diagonalizable over **R** (the real numbers)?

11. If A is an invertible symmetric matrix, what is the relationship between the definiteness of **A** and $\mathbf{A}^{-1}$?

Sketch the curves in Exercises 15 and 19. In each case, draw and label the principal axes, label the intercepts of the curve with the principal axes, and give the formula of the curve in the coordinate system defined by the principal axes.

15. $q(x_1, x_2) = 6x_1^2 + 4x_1x_2 + 3x_2^2 = 1$

19. $q(x_1, x_2) = x_1^2 + 4x_1x_2 + 4x_2^2 = 1$

21. a. Sketch the following three surfaces:

$$\begin{aligned} x_1^2 + 4x_2^2 + 9x_3^2 &= 1 \\ x_1^2 + 4x_2^2 - 9x_3^2 &= 1. \\ -x_1^2 - 4x_2^2 + 9x_3^2 &= 1 \end{aligned}$$

Which of these are bounded? Which are connected? Label the points closest to and farthest from the origin (if there are any).

b. Consider the surface $q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^2 + x_1x_2 + 2x_1x_3 + 3x_2x_3 = 1$.

Which of the three surfaces in part (a) does this surface qualitatively resemble most? Which points on this surface are closest to the origin? Give a rough approximation. You may use technology.