Math 15a – Fall 2007 – Homework #7b

Section 7.5:

Find all the complex eigenvalues of the matrices in Exercises 20, 22, and 24 (including the real ones, of course). Do not use technology. Show all your work.

20. $\mathbf{A} = \begin{bmatrix} 3 & -5 \\ 2 & -3 \end{bmatrix}$ 22. $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -4 & 10 \end{bmatrix}$ 24. $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -7 & 3 \end{bmatrix}$

28. Suppose a 3×3 matrix $\mathbf{A}$ has the real eigenvalue 2 and two complex eigenvalues. Also suppose that $\det(\mathbf{A}) = 50$ and $\mathrm{tr}(\mathbf{A}) = 8$. Find the complex eigenvalues.

30. A real $n \times n$ matrix $\mathbf{A}$ is called a regular transition matrix if all entries of $\mathbf{A}$ are positive, and the entries in each column add up to 1. (See Exercises 24 through 31 of Section 7.2.)

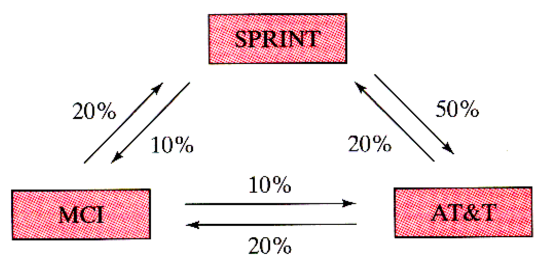
An example is $\mathbf{A} = \begin{bmatrix} 0.4 & 0.3 & 0.1 \\ 0.5 & 0.1 & 0.2 \\ 0.1 & 0.6 & 0.7 \end{bmatrix}$. You may take the following properties of a regular transition matrix for granted (a partial proof is outlined in Exercise 7.2.31.):

- 1 is an eigenvalue of $\mathbf{A}$, with $\dim(E_1) = 1$.
- If λ is a complex eigenvalue of $\mathbf{A}$ other than 1, then $|\lambda| < 1$.

a. Consider a regular $n \times n$ transition matrix $\mathbf{A}$ and a vector $\mathbf{x}$ in $\mathbf{R}^n$ whose entries add up to 1. Show that the entries of $\mathbf{Ax}$ will also add up to 1.

b. Pick a regular transition matrix $\mathbf{A}$, and compute some powers of $\mathbf{A}$ (using technology): $\mathbf{A}^2, \ldots, \mathbf{A}^{10}, \ldots, \mathbf{A}^{100}, \ldots$. What do you observe? Explain your observation. Here, you may assume that there is a complex eigenbasis for $\mathbf{A}$.

32. Most long-distance telephone service in the United States is (perhaps was) provided by three companies: AT&T, MCI, and Sprint. The three companies are in fierce competition, offering discounts or even cash to those who switch. If the figures advertised by the companies are to be believed, people are switching their long-distance provider from one month to the next according to the following diagram:



For example, 20% of the people who use AT&T go to Sprint one month later.

a. We introduce the state vector $\mathbf{x}(t) = \begin{bmatrix} a(t) \\ m(t) \\ s(t) \end{bmatrix}$ fraction using AT&T / fraction using MCI / fraction using Sprint

Find the matrix $\mathbf{A}$ such that $\mathbf{x}(t+1) = \mathbf{Ax}(t)$, assuming that the customer base remains unchanged. Note that $\mathbf{A}$ is a regular transition matrix.

b. Which fraction of the customers will be with each company in the long term? Do you have to know the current market shares to answer this question? Use the power method introduced in Exercise 30.

Section 7.6:

10. Determine whether the zero state is a stable equilibrium of the dynamical system $\mathbf{x}(t+1) = \mathbf{Ax}(t)$,

where $\mathbf{A} = \begin{bmatrix} 0.3 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 \end{bmatrix}$.

12. Given the matrix $\mathbf{A} = \begin{bmatrix} 0.6 & k \\ -k & 0.6 \end{bmatrix}$, for which real numbers k is the zero state a stable equilibrium of the dynamical system $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$?

24. For the matrix $\mathbf{A} = \begin{bmatrix} 1 & -3 \\ 1.2 & -2.6 \end{bmatrix}$, find real closed formulas for the trajectory $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$, where $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Draw a rough sketch.

38. Consider an affine transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, where $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{b}$ is a vector in $\mathbf{R}^n$. (Compare this with Exercise 7.3.45.) Suppose that 1 is not an eigenvalue of $\mathbf{A}$.

a. Find the vector $\mathbf{v}$ in $\mathbf{R}^n$ such that $T(\mathbf{v}) = \mathbf{v}$; this vector is called the equilibrium state of the dynamical system $\mathbf{x}(t+1) = T(\mathbf{x}(t))$.

b. When is the equilibrium $\mathbf{v}$ in part (a) stable (meaning that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{v}$ for all trajectories)?

For additional practice:

Section 7.5:

Find all the complex eigenvalues of the matrices in Exercises 21 and 22 (including the real ones, of course). Do not use technology. Show all your work.

21. $\mathbf{A} = \begin{bmatrix} 11 & -15 \\ 6 & -7 \end{bmatrix}$ 23. $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ 25. $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

27. Suppose a 3×3 matrix $\mathbf{A}$ has only two distinct eigenvalues. Suppose that $\mathrm{tr}(\mathbf{A}) = 1$ and $\det(\mathbf{A}) = 3$. Find the eigenvalues of $\mathbf{A}$ with their algebraic multiplicities.

29. Consider a matrix of the form $\mathbf{A} = \begin{bmatrix} 0 & a & b \\ c & 0 & 0 \\ 0 & d & 0 \end{bmatrix}$ where a, b, c, and d are positive real numbers. Suppose the matrix $\mathbf{A}$ has three distinct real eigenvalues. What can you say about the signs of the eigenvalues? (How many of them are positive, negative, zero?) Is the eigenvalue with the largest absolute value positive or negative?

36. In 1990, the population of the African country Benin was about 4.6 million people. Its composition by age was as follows:

Age Bracket	0-15	15-30	30-45	45-60	60-75	75-90
Percent of Population	46.6	25.7	14.7	8.4	3.8	0.8

We represent these data in a state vector whose components are the populations in the various age brackets, in millions:

$$\mathbf{x}(0) = 4.6 \begin{bmatrix} 0.466 \\ 0.257 \\ 0.147 \\ 0.084 \\ 0.038 \\ 0.008 \end{bmatrix} \approx \begin{bmatrix} 2.14 \\ 1.18 \\ 0.68 \\ 0.39 \\ 0.17 \\ 0.04 \end{bmatrix}$$

We measure time in increments of 15 years, with $t = 0$ in 1990. For example, $\mathbf{x}(3)$ gives the age composition in the year 2035 $(1990 + 3 \cdot 15)$. If current age-dependent birth and death rates are extrapolated, we have the following model:

$$\mathbf{x}(t+1) = \begin{bmatrix} 1.1 & 1.6 & 0.6 & 0 & 0 & 0 \\ 0.82 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.89 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.81 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.53 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.29 & 0 \end{bmatrix} \mathbf{x}(t) = \mathbf{A}\mathbf{x}(t)$$

a. Explain the significance of all the entries in the matrix **A** in terms of population dynamics.

b. Find the eigenvalue of **A** with largest modulus and associated eigenvalue. (Use technology.) What is the significance of these quantities in terms of population dynamics? (For a summary on matrix techniques used in the study of age-structured populations, see Dmitrii O. Logofet, *Matrices and Graphs: Stability Problems in Mathematical Ecology*, Chapters 2 and 3, CRC Press, 1993.)

Section 7.6:

For the matrices in Exercises 3 and 4, determine whether the zero state is a stable equilibrium of the dynamical system $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$.

3. $\mathbf{A} = \begin{bmatrix} 0.8 & 0.7 \\ -0.7 & 0.8 \end{bmatrix}$ 4. $\mathbf{A} = \begin{bmatrix} -0.9 & -0.4 \\ 0.4 & -0.9 \end{bmatrix}$

17. For the matrix $\mathbf{A} = \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}$, find real closed formulas for the trajectory $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$, where $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Draw a rough sketch.

37. Consider the national income of a country, which consists of consumption, investment, and government expenditures. Here we assume the government expenditure to be constant, at G_0, while the national income $Y(t)$, consumption $C(t)$, and investment $I(t)$ change over time. According to a simple model, we have:

$$\left\{\begin{array}{l} Y(t) = C(t) + I(t) + G_0 \\ C(t+1) = \gamma Y(t) \\ I(t+1) = \alpha[C(t+1) - C(t)] \end{array}\right\} \quad \begin{array}{l} (0 < \gamma < 1), \\ (\alpha > 0) \end{array}$$

where γ is the marginal propensity to consume and α is the acceleration coefficient. (See Paul Samuelson, "Interactions between the Multiplier Analysis and the Principle of Acceleration," *Review of Economic Statistics*, May 1939, pp. 75-78.)

a. Find the equilibrium solution of these equations, i.e., when $Y(t+1) = Y(t)$, $C(t+1) = C(t)$, and $I(t+1) = I(t)$.

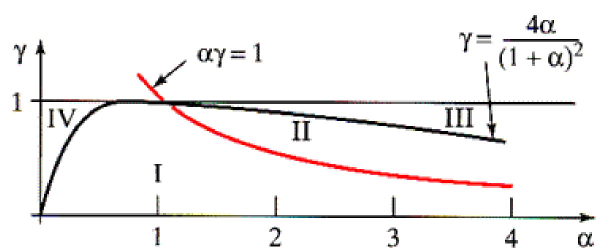
b. Let $y(t)$, $c(t)$, and $i(t)$ be the deviations of $Y(t)$, $C(t)$, and $I(t)$, respectively, from the equilibrium state you found in part (a). These quantities are related by the equations $\left\{\begin{array}{l} y(t) = c(t) + i(t) \\ c(t+1) = \gamma y(t) \\ i(t+1) = \alpha[c(t+1) - c(t)] \end{array}\right\}$. (Verify this!)

By substituting $y(t)$ into the second equation, set up equations of the form $\left\{\begin{array}{l} c(t+1) = p\,c(t) + q\,i(t) \\ i(t+1) = r\,c(t) + s\,i(t) \end{array}\right\}$.

c. When $\alpha = 5$ and $\gamma = 0.2$, determine the stability of the zero state of this system.

d. When $\alpha = 1$ (and γ is arbitrary, $0 < \gamma < 1$), determine the stability of the zero state.

e. For each of the four sectors in the $\alpha\gamma$–plane, determine the stability of the zero state.



Discuss the various cases, in practical terms.

Chapter 7 True/False

1. The eigenvalues of any triangular matrix are its diagonal entries.
2. The trace of any square matrix is the sum of its diagonal entries.
3. The algebraic multiplicity of an eigenvalue cannot exceed its geometric multiplicity.
4. If an $n \times n$ matrix A is diagonalizable (over $\mathbb{R}$), then there must be a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.
5. If the standard vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ are eigenvectors of an $n \times n$ matrix A, then A must be diagonal.
6. If $\vec{v}$ is an eigenvector of A, then $\vec{v}$ must be an eigenvector of A^3 as well.
7. There exists a diagonalizable 5×5 matrix with only two distinct eigenvalues (over $\mathbb{C}$).
8. There exists a real 5×5 matrix without any real eigenvalues.
9. If 0 is an eigenvalue of a matrix A, then $\det(A) = 0$.
10. The eigenvalues of a 2×2 matrix A are the solutions of the equation $\lambda^2 - (\operatorname{tr} A)\lambda + (\det A) = 0$.
11. If 1 is the only eigenvalue of an $n \times n$ matrix A, then A must be I_n.
12. If A and B are $n \times n$ matrices, if α is an eigenvalue of A, and if β is an eigenvalue of B, then $\alpha\beta$ must be an eigenvalue of AB.
13. If 3 is an eigenvalue of an $n \times n$ matrix A, then 9 must be an eigenvalue of A^2.
14. The matrix of any orthogonal projection onto a subspace V of $\mathbb{R}^n$ is diagonalizable.
15. If matrices A and B have the same eigenvalues (over $\mathbb{C}$), with the same algebraic multiplicities, then matrices A and B must have the same trace.
16. If a real matrix A has only the eigenvalues 1 and -1, then A must be orthogonal.
17. Any rotation-scaling matrix in $\mathbb{R}^{2\times 2}$ is diagonalizable over $\mathbb{C}$.
18. If A is a noninvertible $n \times n$ matrix, then the geometric multiplicity of eigenvalue 0 is $n - \operatorname{rank}(A)$.
19. If matrix A is diagonalizable, then its transpose A^T must be diagonalizable as well.
20. If A and B are two 3×3 matrices such that $\operatorname{tr}(A) = \operatorname{tr}(B)$ and $\det(A) = \det(B)$, then A and B must have the same eigenvalues.
21. If matrix A^2 is diagonalizable, then matrix A must be diagonalizable as well.
22. The determinant of a matrix is the product of its eigenvalues (over $\mathbb{C}$), counted with their algebraic multiplicities.
23. All lower triangular matrices are diagonalizable (over $\mathbb{C}$).
24. If two $n \times n$ matrices A and B are diagonalizable, then AB must be diagonalizable as well.
25. If an invertible matrix A is diagonalizable, then A^{-1} must be diagonalizable as well.
26. If $\det(A) = \det(A^T)$, then matrix A must be symmetric.
27. If matrix $A = \begin{bmatrix} 7 & a & b \\ 0 & 7 & c \\ 0 & 0 & 7 \end{bmatrix}$ is diagonalizable, then a, b, and c must all be zero.
28. If two $n \times n$ matrices A and B are diagonalizable, then $A + B$ must be diagonalizable as well.
29. All diagonalizable matrices are invertible.
30. If vector $\vec{v}$ is an eigenvector of both A and B, then $\vec{v}$ must be an eigenvector of $A + B$.
31. If an $n \times n$ matrix A is diagonalizable, then A must have n distinct eigenvalues.
32. If two 3×3 matrices A and B both have the eigenvalues 1, 2, and 3, then A must be similar to B.
33. If $\vec{v}$ is an eigenvector of A, then $\vec{v}$ must be an eigenvector of A^T as well.
34. All invertible matrices are diagonalizable.
35. If $\vec{v}$ and $\vec{w}$ are linearly independent eigenvectors of matrix A, then $\vec{v} + \vec{w}$ must be an eigenvector of A as well.
36. If a 2×2 matrix R represents a reflection about a line L, then R must be diagonalizable.
37. If A is a 2×2 matrix such that $\operatorname{tr}(A) = 1$ and $\det(A) = -6$, then A must be diagonalizable.
38. If a matrix is diagonalizable, then the algebraic multiplicity of each of its eigenvalues λ must equal the geometric multiplicity of λ.
39. If $\vec{u}, \vec{v}, \vec{w}$ are eigenvectors of a 4×4 matrix A, with associated eigenvalues 3, 7, and 11, respectively, then vectors $\vec{u}, \vec{v}, \vec{w}$ must be linearly independent.
40. If a 4×4 matrix A is diagonalizable, then the matrix $A + 4I_4$ must be diagonalizable as well.
41. All orthogonal matrices are diagonalizable (over $\mathbb{R}$).
42. If A is an $n \times n$ matrix and λ is an eigenvalue of the partitioned matrix $M = \begin{bmatrix} A & A \\ 0 & A \end{bmatrix}$, then λ must be an eigenvalue of matrix A.
43. If two matrices A and B have the same characteristic polynomials, then they must be similar.
44. If A is a diagonalizable 4×4 matrix with $A^4 = 0$, then A must be the zero matrix.
45. If an $n \times n$ matrix A is diagonalizable (over $\mathbb{R}$), then every vector $\vec{v}$ in $\mathbb{R}^n$ can be expressed as a sum of eigenvectors of A.

46. If vector $\vec{v}$ is an eigenvector of both A and B, then $\vec{v}$ is an eigenvector of AB.

47. Similar matrices have the same characteristic polynomials.

48. If a matrix A has k distinct eigenvalues, then $\text{rank}(A) \geq k$.

49. If the rank of a square matrix A is 1, then all the nonzero vectors in the image of A are eigenvectors of A.

50. If the rank of an $n \times n$ matrix A is 1, then A must be diagonalizable.

51. If A is a 4×4 matrix with $A^4 = 0$, then 0 is the only eigenvalue of A.

52. If two $n \times n$ matrices A and B are both diagonalizable, then they must commute.

53. If $\vec{v}$ is an eigenvector of A, then $\vec{v}$ must be in the kernel of A or in the image of A.

54. All symmetric 2×2 matrices are diagonalizable (over $\mathbb{R}$).

55. If A is a 2×2 matrix with eigenvalues 3 and 4 and if $\vec{u}$ is a unit eigenvector of A, then the length of vector $A\vec{u}$ cannot exceed 4.

56. If $\vec{u}$ is a nonzero vector in $\mathbb{R}^n$, then $\vec{u}$ must be an eigenvector of matrix $\vec{u}\vec{u}^T$.

57. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is an eigenbasis for both A and B, then matrices A and B must commute.

58. If $\vec{v}$ is an eigenvector of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\vec{v}$ must be an eigenvector of its classical adjoint $\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ as well.