Math 15a – Fall 2007 – Homework #5b

Section 5.3:

If the $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are orthogonal matrices, which of the matrices in Exercises 5 through 11 must be orthogonal as well?

5. $3\mathbf{A}$ 6. $-\mathbf{B}$ 7. $\mathbf{AB}$ 8. $\mathbf{A}+\mathbf{B}$ 9. $\mathbf{B}^{-1}$ 10. $\mathbf{B}^{-1}\mathbf{AB}$ 11. $\mathbf{A}^{T}$

40. Consider the subspace W of $\mathbf{R}^4$ spanned by the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 9 \\ -5 \\ 3 \end{bmatrix}$.

    Find the matrix of the orthogonal projection onto W.

42. Let $\mathbf{A}$ be the matrix of an orthogonal projection. Find $\mathbf{A}^2$ in two ways:

    a. Geometrically. (Consider what happens when you apply an orthogonal projection twice.)

    b. By computation, using the formula given in Fact 5.3.10 (matrix of an orthogonal projection in terms of an orthonormal basis for a given subspace).

44. Consider an $n \times m$ matrix $\mathbf{A}$. Find $\dim(\text{im}(\mathbf{A})) + \dim(\ker(\mathbf{A}^T))$, in terms of m and n.

46. Consider a QR-factorization $\mathbf{M} = \mathbf{QR}$. Show that $\mathbf{R} = \mathbf{Q}^T\mathbf{M}$.

47. If $\mathbf{A} = \mathbf{QR}$ is a QR-factorization, what is the relationship between $\mathbf{A}^T\mathbf{A}$ and $\mathbf{R}^T\mathbf{R}$?

68. The formula $\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$ for the matrix of an orthogonal projection is derived in Exercise 67 [and in section 5.4 and in class]. Now consider the **QR** factorization of A, and express the matrix $\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$ in terms of $\mathbf{Q}$.

Section 5.4:

2. Consider the subspace im($\mathbf{A}$) of $\mathbf{R}^3$, where $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$. Find a basis of $\ker(\mathbf{A}^T)$, and draw a sketch illustrating the formula $(\text{im } \mathbf{A})^\perp = \ker(\mathbf{A}^T)$ in this case.

4. Let $\mathbf{A}$ be an $n \times m$ matrix. Is the formula $(\ker \mathbf{A})^\perp = \text{im}(\mathbf{A}^T)$ necessarily true? Explain.

5. Let V be the solution space of the linear system $\left\{\begin{matrix} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + 2x_2 + 5x_3 + 4x_4 = 0 \end{matrix}\right\}$. Find a basis for $V^\perp$.

16. Use the formula $(\text{im } \mathbf{A})^\perp = \ker(\mathbf{A}^T)$ to prove the equation $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T)$.

20. By using paper and pencil, find the least-squares solution $\mathbf{x}^*$ of the system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$. Verify that the vector $\mathbf{b} - \mathbf{Ax}^*$ is perpendicular to the image of $\mathbf{A}$.

26. Find the least-squares solution $\mathbf{x}^*$ of the system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

32. Fit a quadratic polynomial to the data points (0, 27), (1, 0), (2, 0), (3, 0), using least squares. Sketch the solution.

38. In the accompanying table, we list the height h, the gender g, and the weight w of some young adults.

Height h (in inches above 5 feet)	**Gender g** (1 = "female", 0 = "male")	**Weight w** (in pounds)
2	1	110
12	0	180
5	1	120
11	1	160
6	0	160

Fit a function of the form $w = c_0 + c_1 h + c_2 g$ to these data, using least squares. Before you do the computations, think about the signs of c_1 and c_2.What signs would you expect if these data were representative of the general population? Why? What is the sign of c_0? What is the practical significance of c_0?

For additional practice:
Section 5.3:

31. Are the *rows* of an orthogonal matrix **A** necessarily orthonormal?

32. (a) Consider an $n \times m$ matrix **A** such that $\mathbf{A}^T\mathbf{A} = \mathbf{I}_m$. Is it necessarily true that $\mathbf{A}\mathbf{A}^T = \mathbf{I}_n$? Explain.

(b) Consider an $n \times n$ matrix **A** such that $\mathbf{A}^T\mathbf{A} = \mathbf{I}_m$. Is it necessarily true that $\mathbf{A}\mathbf{A}^T = \mathbf{I}_n$? Explain.

37. Is there an orthogonal transformation T from $\mathbf{R}^3$ to $\mathbf{R}^3$ such that $T\begin{bmatrix}2\\3\\0\end{bmatrix} = \begin{bmatrix}3\\0\\2\end{bmatrix}$ and $T\begin{bmatrix}-3\\2\\0\end{bmatrix} = \begin{bmatrix}2\\-3\\0\end{bmatrix}$?

45. For which $n \times m$ matrices **A** does the equation $\dim(\ker(\mathbf{A})) = \dim(\ker(\mathbf{A}^T))$ hold? Explain.

48. Consider an invertible $n \times n$ matrix **A**. Can you write **A** as $\mathbf{A} = \mathbf{LQ}$, where **L** is a lower triangular matrix and **Q** is orthogonal? *Hint*: Consider the QR factorization of $\mathbf{A}^T$.

Section 5.4:

1. Consider the subspace im(**A**) of $\mathbf{R}^2$, where $\mathbf{A} = \begin{bmatrix}2 & 4\\3 & 6\end{bmatrix}$. Find a basis of $\ker(\mathbf{A}^T)$, and draw a sketch illustrating the formula $(\text{im } \mathbf{A})^\perp = \ker(\mathbf{A}^T)$ in this case.

6. If **A** is an $n \times m$ matrix, is the formula $\text{im}(\mathbf{A}) = \text{im}(\mathbf{A}\mathbf{A}^T)$ necessarily true? Explain.

7. Consider a symmetric $n \times n$ matrix **A**. What is the relationship between im(**A**) and ker(**A**)?

15. Consider an $m \times n$ matrix **A** with $\ker(\mathbf{A}) = \{\mathbf{0}\}$. Show that there exists an $n \times m$ matrix **B** such that $\mathbf{BA} = \mathbf{I}_n$. *Hint*: $\mathbf{A}^T\mathbf{A}$ is invertible.

17. Does the equation $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T\mathbf{A})$ hold for all $n \times m$ matrices **A**? Explain.

18. Does the equation $\text{rank}(\mathbf{A}^T\mathbf{A}) = \text{rank}(\mathbf{A}\mathbf{A}^T)$ hold for all $n \times m$ matrices **A**? Explain. *Hint*: Exercise 17 is useful.

22. Find the least-squares solution $\mathbf{x}^*$ of the system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = \begin{bmatrix}3 & 2\\5 & 3\\4 & 5\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}5\\9\\2\end{bmatrix}$.

Determine the error $\left\|\mathbf{b} - \mathbf{A}\mathbf{x}^*\right\|$.

24. Find the least-squares solution $\mathbf{x}^*$ of the system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}3\\2\\7\end{bmatrix}$. Draw a sketch showing the vector **b**, the image of **A**, the vector $\mathbf{A}\mathbf{x}^*$, and the vector $\mathbf{b} - \mathbf{A}\mathbf{x}^*$.

31. Fit a linear function of the form $f(t) = c_0 + c_1 t$ to the data points (0, 3), (1, 3), (1, 6), using least squares. Sketch the solution. [Note: Strictly speaking, a function of this form is not a linear function in the sense that we use in this course. More properly, this might be called an "affine" function.]

37. The accompanying table lists several commercial airlines, the year they were introduced, and the number of displays in the cockpit.

Plane	**Year t**	**Displays d**
Douglas DC-3	1935	35
Lockheed Constellation	1946	46
Boeing 707	1959	77
Concorde	1969	133

a. Fit a linear function of the form $\log(d) = c_0 + c_1 t$ to the data points $(t_i, \log(d_i))$, using least squares.

b. Use your answer in part (a) to fit an exponential function $d = ka^t$ to the data points (t_i, d_i).

c. The Airbus A320 was introduced in 1988. Based on your answer in part (b), how many displays do you expect in the cockpit of this plane? (There are 93 displays in the cockpit of an Airbus A320.) Explain.

41. In the accompanying table, we list the public debt D of the United States (in billions of dollars), in the year t (as of September 30).

t	1970	1975	1980	1985	1990	1995
D	370	533	908	1823	3233	4974

a. Fit a linear function of the form $\log(D) = c_0 + c_1 t$ to the data points $(t_i, \log(D_i))$, using least squares. Use the result to fit an exponential function to the data points (t_i, D_i).

b. What debt does your formula in part (a) predict for the year 2000? What about the year 2010?

c. On Sept 30, 2000, the debt was 5,674 billion dollars. What happened?

42. If **A** is any matrix, show that the linear transformation $L(\mathbf{x}) = \mathbf{Ax}$ from im($\mathbf{A}^T$) to im(**A**) is an isomorphism. This provides yet another proof of the formula rank(**A**) = rank($\mathbf{A}^T$).

Chapter 5 True/False Exercises

1. If matrix A is orthogonal, then matrix A^2 must be orthogonal as well.
2. The equation $(AB)^T = A^T B^T$ holds for all $n \times n$ matrices A and B.
3. If A and B are symmetric $n \times n$ matrices, then $A + B$ must be symmetric as well.
4. If matrices A and S are orthogonal, then $S^{-1}AS$ is orthogonal as well.
5. All nonzero symmetric matrices are invertible.
6. If A is an $n \times n$ matrix such that $AA^T = I_n$, then A must be an orthogonal matrix.
7. If $\vec{u}$ is a unit vector in $\mathbb{R}^n$, and $L = \text{span}(\vec{u})$, then $\text{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{x}$ for all vectors $\vec{x}$ in $\mathbb{R}^n$.
8. If A is a symmetric matrix, then $7A$ must be symmetric as well.
9. If T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n)$ are all unit vectors, then T must be an orthogonal transformation.
10. If A is an invertible matrix, then the equation $(A^T)^{-1} = (A^{-1})^T$ must hold.
11. If A and B are symmetric $n \times n$ matrices, then $ABBA$ must be symmetric as well.
12. If matrices A and B commute, then matrices A^T and B^T must commute as well.
13. There exists a subspace V of $\mathbb{R}^5$ such that $\dim(V) = \dim(V^\perp)$, where $V^\perp$ denotes the orthogonal complement of V.
14. Every invertible matrix A can be expressed as the product of an orthogonal matrix and an upper triangular matrix.
15. If $\vec{x}$ and $\vec{y}$ are two vectors in $\mathbb{R}^n$, then the equation $\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$ must hold.
16. The equation $\det(A^T) = \det(A)$ holds for all 2×2 matrices A.
17. If matrix A is orthogonal, then A^T must be orthogonal as well.
18. If A and B are symmetric $n \times n$ matrices, then AB must be symmetric as well.
19. If matrices A and B commute, then A must commute with B^T as well.
20. If A is any matrix with $\ker(A) = \{\vec{0}\}$, then the matrix AA^T represents the orthogonal projection onto the image of A.
21. The entries of an orthogonal matrix are all less than or equal to 1.
22. Every nonzero subspace of $\mathbb{R}^n$ has an orthonormal basis.
23. $\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$ is an orthogonal matrix.
24. If V is a subspace of $\mathbb{R}^n$ and $\vec{x}$ is a vector in $\mathbb{R}^n$, then vector $\text{proj}_V \vec{x}$ must be orthogonal to vector $\vec{x} - \text{proj}_V \vec{x}$.
25. If A and B are orthogonal 2×2 matrices, then $AB = BA$.
26. If A is a symmetric matrix, vector $\vec{v}$ is in the image of A, and $\vec{w}$ is in the kernel of A, then the equation $\vec{v} \cdot \vec{w} = 0$ must hold.
27. The formula $\ker(A) = \ker(A^T A)$ holds for all matrices A.

28. If $A^T A = AA^T$ for an $n \times n$ matrix A, then A must be orthogonal.

29. The determinant of all orthogonal 2×2 matrices is 1.

30. If A is any square matrix, then matrix $\frac{1}{2}(A - A^T)$ is skew-symmetric.

31. If A is an invertible matrix such that $A^{-1} = A$, then A must be orthogonal.

32. If the entries of two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ are all positive, then $\vec{v}$ and $\vec{w}$ must enclose an acute angle.

33. The formula $(\ker B)^{\perp} = \operatorname{im}(B^T)$ holds for all matrices B.

34. The matrix $A^T A$ is symmetric for all matrices A.

35. If matrix A is similar to B and A is orthogonal, then B must be orthogonal as well.

36. The formula $\operatorname{im}(B) = \operatorname{im}(B^T B)$ holds for all square matrices B.

37. If matrix A is symmetric and matrix S is orthogonal, then matrix $S^{-1}AS$ must be symmetric.

38. If A is a square matrix such that $A^T A = AA^T$, then $\ker(A) = \ker(A^T)$.

39. There exist orthogonal 2×2 matrices A and B such that $A + B$ is orthogonal as well.

40. If $\|A\vec{x}\| \leq \|\vec{x}\|$ for all $\vec{x}$ in $\mathbb{R}^n$, then A must represent the orthogonal projection onto a subspace V of $\mathbb{R}^n$.

41. Any square matrix can be written as the sum of a symmetric and a skew-symmetric matrix.

42. If $x_1, x_2, \ldots, x_n$ are any real numbers, then the inequality

$$\left(\sum_{k=1}^{n} x_k\right)^2 \leq n \sum_{k=1}^{n} (x_k^2)$$

must hold.

43. If $AA^T = A^2$ for a 2×2 matrix A, then A must be symmetric.

44. If V is a subspace of $\mathbb{R}^n$ and $\vec{x}$ is a vector in $\mathbb{R}^n$, then the inequality $\vec{x} \cdot (\operatorname{proj}_V \vec{x}) \geq 0$ must hold.

45. If A is an $n \times n$ matrix such that $\|A\vec{u}\| = 1$ for all unit vectors $\vec{u}$, then A must be an orthogonal matrix.

46. If A is any symmetric 2×2 matrix, then there must exist a real number x such that matrix $A - xI_2$ fails to be invertible.

47. There exists a basis of $\mathbb{R}^{2\times 2}$ that consists of orthogonal matrices.

48. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, then the matrix Q in the QR-factorization of A is a rotation matrix.

49. There exists a linear transformation L from $\mathbb{R}^{3\times 3}$ to $\mathbb{R}^{2\times 2}$ whose kernel is the space of all skew-symmetric 3×3 matrices.

50. If a 3×3 matrix A represents the orthogonal projection onto a plane V in $\mathbb{R}^3$, then there must exist an orthogonal 3×3 matrix S such that $S^T AS$ is diagonal.