Math 15a – Fall 2007 – Homework #4

Problems due Mon, Oct 22:

Section 4.1:

Find a basis for each of the spaces in Exercises 20, 26, and 30 and determine its dimension.

20. The space of all matrices $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathbf{R}^{2\times 2}$ such that $a + d = 0$.

26. The space of all polynomials $f(t)$ in P_3 such that $f(1) = 0$ and $\int_{-1}^{1} f(t)\,dt = 0$.

30. The space of all 2×2 matrices $\mathbf{A}$ such that $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Section 4.2:

6. Is the transformation $T(\mathbf{M}) = \mathbf{M}\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ from $\mathbf{R}^{2\times 2}$ to $\mathbf{R}^{2\times 2}$ linear? If it is, determine whether it is an isomorphism. [*Hint*: To show that T is isomorphism, it's enough to show that ker(T) consists only of the zero element in this space. If ker(T) is nonzero, then T cannot be an isomorphism. See Ex. 53 below.]

25. Is the transformation $[T(f)](t) = f''(t) + 4f'(t)$ from P_2 to P_2 linear? If it is, determine whether it is an isomorphism.

52. Find the kernel and nullity of the transformation in Exercise 6.

53. Find the image, rank, kernel and nullity of the transformation in Exercise 25.

66. Find the kernel and nullity of the transformation $T(f) = f - f'$ from C^∞ to C^∞. [C^∞ denotes the linear space consisting of all infinitely differentiable functions of one variable.]

Section 4.3:

22. Find the matrix of the linear transformation $T(f) = f'' + 4f'$ from P_2 to P_2 relative to the basis $\mathbb{U} = \{1, t, t^2\}$.

27. Find the matrix of the linear transformation $[T(f)](t) = f(2t-1)$ from P_2 to P_2 relative to the basis $\mathbb{U} = \{1, t, t^2\}$.

28. Find the matrix of the linear transformation $[T(f)](t) = f(2t-1)$ from P_2 to P_2 relative to the basis $\mathbb{B} = \{1, t-1, (t-1)^2\}$.

47. a. Find the change of basis matrix $\mathbf{S}$ from the basis $\mathbb{B}$ considered in Exercise 28 to the standard basis $\mathbb{U} = \{1, t, t^2\}$ of P_2 considered in Exercise 27.

    b. Verify the formula $\mathbf{SB} = \mathbf{AS}$ (that is, $\mathbf{B} = \mathbf{S}^{-1}\mathbf{AS}$) for the matrices $\mathbf{B}$ and $\mathbf{A}$ you found in Exercises 28 and 27, respectively.

    c. Find the change of basis matrix from $\mathbb{U}$ to $\mathbb{B}$.

For additional practice:

Section 4.1:

Which of the subsets of P_2 given in Exercises 1, 2, and 3 are subspaces of P_2? Find a basis for those that are subspaces. [P_2 is the linear space consisting of polynomials of degree less than or equal to 2.]

1. $\{p(t) : p(0) = 2\}$ 2. $\{p(t) : p(0) = 0\}$ 3. $\{p(t) : p'(1) = p(2)\}$ (p' denotes the derivative.)

Which of the subsets of $\mathbf{R}^{3\times 3}$ such given in Exercises 9, 10, and 11 are subspaces of $\mathbf{R}^{3\times 3}$?

9. The 3×3 matrices whose entries are all greater than or equal to zero.

10. The 3×3 matrices **A** such that the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in the kernel of **A**.

11. The 3×3 matrices in reduced row-echelon form.

Find a basis for each of the spaces in Exercises 25 and 29 and determine its dimension.

25. The space of all polynomials $f(t)$ in P_2 such that $f(1) = 0$.

29. The space of all 2×2 matrices **A** such that $\mathbf{A}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Section 4.2:

2. Is the transformation $T(\mathbf{M}) = 7\mathbf{M}$ from $\mathbf{R}^{2\times 2}$ to $\mathbf{R}^{2\times 2}$ linear? If so, determine whether it is an isomorphism.

4. Is the transformation $T(\mathbf{M}) = \det(\mathbf{M})$ from $\mathbf{R}^{2\times 2}$ to $\mathbf{R}$ linear? If so, determine whether it is an isomorphism.

67. For which constants k is the linear transformation $T(\mathbf{M}) = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}\mathbf{M} - \mathbf{M}\begin{bmatrix} 3 & 0 \\ 0 & k \end{bmatrix}$ an isomorphism from $\mathbf{R}^{2\times 2}$ to $\mathbf{R}^{2\times 2}$.

81. In this exercise, we will outline a proof of the Rank-Nullity Theorem: If T is a linear transformation from V to W, where V is finite-dimensional, then $\dim(V) = \dim(\text{im } T) + \dim(\ker T) = \text{rank}(T) + \text{nullity}(T)$.

    a. Explain why $\ker(T)$ and $\text{image}(T)$ are finite dimensional. *Hint*: Use Exercises 4.1.54 and 4.1.57.

    Now, consider a basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of $\ker(T)$, where $n = \text{nullity}(T)$, and a basis $\{\mathbf{w}_1, \mathbf{v}_2, \ldots, \mathbf{w}_r\}$ of $\text{im}(T)$, where $r = \text{rank}(T)$. Consider vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$ in V such that $T(\mathbf{u}_i) = \mathbf{w}_i$ for $i = 1, \ldots, r$. Our goal is to show that the $r + n$ vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ form a basis of V. This will prove our claim.

    b. Show that the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent. *Hint*: Consider a relation $c_1\mathbf{u}_1 + \cdots c_r\mathbf{u}_r + d_1\mathbf{v}_1 + \cdots d_n\mathbf{v}_n = \mathbf{0}$, apply linear transformation T to both sides, and take it from there.

    c. Show that the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ span V. *Hint*: Consider an arbitrary vector $\mathbf{v}$ in V, and write $T(\mathbf{v}) = d_1\mathbf{w}_1 + \cdots d_r\mathbf{w}_r$. Now show that the vector $\mathbf{v} - d_1\mathbf{u}_1 + \cdots d_r\mathbf{u}_r$ is in the kernel of T, so that $\mathbf{v} - d_1\mathbf{u}_1 + \cdots d_r\mathbf{u}_r$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Section 4.3:

1. Are the polynomials $f(t) = 7 + 3t + t^2$, $g(t) = 9 + 9t + 4t^2$, and $h(t) = 3 + 2t + t^2$ linearly independent?

13. Find the matrix of the linear transformation $T(\mathbf{M}) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\mathbf{M}$ from $\mathbf{R}^{2\times 2}$ to $\mathbf{R}^{2\times 2}$ with respect to the basis

    $\mathbb{U} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$.

14. Find the matrix of the linear transformation $T(\mathbf{M}) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\mathbf{M}$ from $\mathbf{R}^{2\times 2}$ to $\mathbf{R}^{2\times 2}$ with respect to the basis

    $\mathbb{B} = \left\{\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\right\}$.

44. a. Find the change of basis matrix **S** from the basis $\mathbb{B}$ considered in Exercise 14 to the standard basis $\mathbb{U}$ of $\mathbf{R}^{2\times 2}$ considered in Exercise 13.

    b. Verify the formula **SB** = **AS** (that is, $\mathbf{B} = \mathbf{S}^{-1}\mathbf{AS}$) for the matrices **B** and **A** you found in Exercises 14 and 13, respectively.