Math 15a – Fall 2007 – Homework #3b

Section 3.3:

30. Find a basis of the subspace of $\mathbf{R}^4$ defined by the equation $2x_1 - x_2 + 2x_3 + 4x_4 = 0$.

32. Find a basis of the subspace of $\mathbf{R}^4$ that consists of all vectors perpendicular to both $\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\\2\\3\end{bmatrix}$.

36. Can you find a 3 × 3 matrix **A** such that im(**A**) = ker(**A**)? Explain.

Section 3.4:

In Exercises 6, 8, and 18, determine whether the vector **x** is in the span *V* of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ (proceed "by inspection" if possible, and use the reduced row-echelon form if necessary). If **x** is in *V*, find the coordinates of **x** with respect to the basis $\mathbb{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ of *V*, and write the coordinate vector $[\mathbf{x}]_{\mathbb{B}}$.

6. $\mathbf{x} = \begin{bmatrix}-4\\4\end{bmatrix};\ \mathbf{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}5\\6\end{bmatrix}$

8. $\mathbf{x} = \begin{bmatrix}2\\3\\4\end{bmatrix};\ \mathbf{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}2\\0\\1\end{bmatrix}$

18. $\mathbf{x} = \begin{bmatrix}5\\4\\3\\2\end{bmatrix};\ \mathbf{v}_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix},\ \mathbf{v}_3 = \begin{bmatrix}0\\-1\\0\\1\end{bmatrix}$

In Exercises 26 and 28, find the matrix **B** of the linear transformation T(**x**) = **Ax** with respect to the basis $\mathbb{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$.

26. $\mathbf{A} = \begin{bmatrix}0 & 1\\2 & 3\end{bmatrix};\ \mathbf{v}_1 = \begin{bmatrix}1\\2\end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix}1\\1\end{bmatrix}$

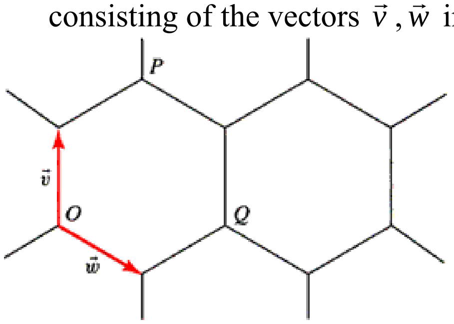
28. $\mathbf{A} = \begin{bmatrix}5 & -4 & -2\\-4 & 5 & -2\\-2 & -2 & 8\end{bmatrix};\ \mathbf{v}_1 = \begin{bmatrix}2\\2\\1\end{bmatrix};\ \mathbf{v}_2 = \begin{bmatrix}1\\-1\\0\end{bmatrix};\ \mathbf{v}_3 = \begin{bmatrix}0\\1\\-2\end{bmatrix}$

42. Find a basis $\mathbb{B}$ of $\mathbf{R}^3$ such that the $\mathbb{B}$-matrix **B** of the linear transformation given by reflection *T* about the plane $x_1 - 2x_2 + x_3 = 0$ in $\mathbf{R}^3$ is diagonal.

44. Consider the plane $2x_1 - 3x_2 + 4x_3 = 0$ with basis $\mathbb{B}$ consisting of vectors $\begin{bmatrix}8\\4\\-1\end{bmatrix}$ and $\begin{bmatrix}5\\2\\-1\end{bmatrix}$. If $\mathbf{x}_{\mathbb{B}} = \begin{bmatrix}2\\-1\end{bmatrix}$, find **x**.

46. Consider the plane $x_1 + 2x_2 + x_3 = 0$. Find a basis $\mathbb{B}$ of this plane such that $\mathbf{x}_{\mathbb{B}} = \begin{bmatrix}2\\-1\end{bmatrix}$ for $\mathbf{x} = \begin{bmatrix}1\\-1\\1\end{bmatrix}$.

50. Given a hexagonal tiling of the plane, such as you might find on a kitchen floor, consider the basis $\mathbb{B}$ of $\mathbf{R}^2$ consisting of the vectors $\vec{v}, \vec{w}$ in the following sketch:



a. Find the coordinate vectors $\left[\overrightarrow{OP}\right]_{\mathbb{B}}$ and $\left[\overrightarrow{OQ}\right]_{\mathbb{B}}$.
Hint: Sketch the coordinate grid defined by the basis $\mathbb{B} = \{\vec{v}, \vec{w}\}$.

b. We are told that $\left[\overrightarrow{OR}\right]_{\mathbb{B}} = \begin{bmatrix}3\\2\end{bmatrix}$. Sketch the point *R*. Is *R* the vertex or the center of a tile?

c. We are told that $\left[\overrightarrow{OS}\right]_{\mathbb{B}} = \begin{bmatrix}17\\13\end{bmatrix}$. Is *S* the center or the vertex of a tile?

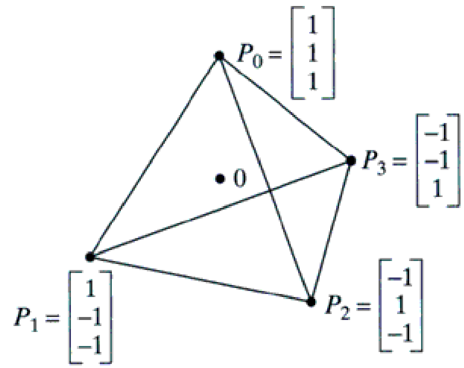
56. Find a basis $\mathbb{B}$ of $\mathbf{R}^2$ such that $\begin{bmatrix}1\\2\end{bmatrix}_{\mathbb{B}} = \begin{bmatrix}3\\5\end{bmatrix}$ and $\begin{bmatrix}3\\4\end{bmatrix}_{\mathbb{B}} = \begin{bmatrix}2\\3\end{bmatrix}$.

74. Consider the regular tetrahedron in the accompanying sketch whose center is at the origin. Let $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the position vectors of the four vertices of the tetrahedron: $\mathbf{v}_0 = OP_0$, $\mathbf{v}_1 = OP_1$, $\mathbf{v}_2 = OP_2$, and $\mathbf{v}_3 = OP_3$.

a. Find the sum $\mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$.

b. Find the coordinate vector of $\mathbf{v}_0$ with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

c. Let T be the linear transformation with $T(\mathbf{v}_0) = \mathbf{v}_3$, $T(\mathbf{v}_3) = \mathbf{v}_1$, and $T(\mathbf{v}_1) = \mathbf{v}_0$. What is $T(\mathbf{v}_2)$? Describe the transformation T geometrically (as a reflection, rotation, projection, or whatever). Find the matrix $\mathbf{B}$ of T with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. What is $\mathbf{B}^3$? Explain.



Note: The same figure was used in Exercise 2.4.48.

For additional practice:

Section 3.3:

27. Determine whether the vectors $\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\2\\4\\8\end{bmatrix}, \begin{bmatrix}1\\-2\\4\\-8\end{bmatrix}\right\}$ form a basis of $\mathbf{R}^4$.

29. Find a basis of the subspace of $\mathbf{R}^3$ defined by the equation $2x_1 + 3x_2 + x_3 = 0$.

40. Consider two subspaces V and W of $\mathbf{R}^n$, where V is contained in W. Explain why $\dim(V) \le \dim(W)$. (This statement seems intuitively rather obvious. Still, we cannot rely on our intuition when dealing with $\mathbf{R}^n$.)

41. Consider two subspaces V and W of $\mathbf{R}^n$, where V is contained in W. In Exercise 40, we learned that $\dim(V) \le \dim(W)$. Show that if $\dim(V) = \dim(W)$, then $V = W$.

42. Consider a subspace V of $\mathbf{R}^n$ with $\dim(V) = n$. Explain why $V = \mathbf{R}^n$.

61. Consider a 4×2 matrix $\mathbf{A}$ and a 2×5 matrix $\mathbf{B}$.

a. What are the possible dimensions of the *kernel* of $\mathbf{AB}$?

b. What are the possible dimensions of the *image* of $\mathbf{AB}$?

Section 3.4:

In Exercises 5, 7, 17, and 18, determine whether the vector $\mathbf{x}$ is in the span V of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ (proceed "by inspection" if possible, and use the reduced row-echelon form if necessary). If $\mathbf{x}$ is in V, find the coordinates of $\mathbf{x}$ with respect to the basis $\mathbb{R} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ of V, and write the coordinate vector $[\mathbf{x}]_{\mathbb{B}}$.

5. $\mathbf{x} = \begin{bmatrix}7\\16\end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix}2\\5\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}5\\12\end{bmatrix}$

7. $\mathbf{x} = \begin{bmatrix}3\\1\\-4\end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix}1\\-1\\0\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}0\\1\\-1\end{bmatrix}$

17. $\mathbf{x} = \begin{bmatrix}1\\1\\1\\-1\end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix}1\\0\\2\\0\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}0\\1\\3\\0\end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix}0\\0\\4\\1\end{bmatrix}$

27. Find the matrix **B** of the linear transformation T(**x**) = **Ax** with respect to the

basis $\mathbb{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ where $\mathbf{A} = \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$; $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$; $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

45. Consider the plane $2x_1 - 3x_2 + 4x_3 = 0$ Find a basis $\mathbb{B}$ of this plane such that $\mathbf{x}_{\mathbb{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ for $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.

55. Consider the basis $\mathbb{B}$ of $\mathbf{R}^2$ consisting of the vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and let $\mathbb{R}$ be the basis consisting

of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Find a matrix **P** such that $[\mathbf{x}]_{\mathbb{R}} = \mathbf{P}[\mathbf{x}]_{\mathbb{B}}$.