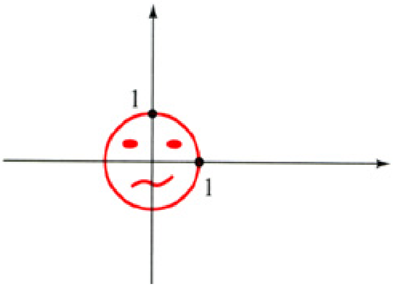
Math 15a – Fall 2007 – Homework #2

Section 2.1:

8. Find the inverse of the linear transformation $\begin{Bmatrix} y_1 = x_1 + 7x_2 \\ y_2 = 3x_1 + 20x_2 \end{Bmatrix}$. [That is, solve for x_1, x_2 in terms of y_1, y_2.]

Consider the circular face in the accompanying figure. For each of the matrices **A** in Exercises 24 through 30, draw a sketch showing the effect of the linear transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ on this face.



24. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ 25. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ 26. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ 27. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

28. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ 29. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ 30. $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

43. a. Consider the vector $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.

Is the transformation $T(\mathbf{x}) = \mathbf{v} \cdot \mathbf{x}$ (the dot product) from $\mathbf{R}^3$ to $\mathbf{R}$ linear? If so, find the matrix of T.

b. Consider an arbitrary vector **v** in $\mathbf{R}^3$. Is the transformation $T(\mathbf{x}) = \mathbf{v} \cdot \mathbf{x}$ linear?
If so, find the matrix of T (in terms of the components of **v**.

c. Conversely, consider a linear transformation T from $\mathbf{R}^3$ to **R**.
Show that there exists a vector **v** in $\mathbf{R}^3$ such that $T(\mathbf{x}) = \mathbf{v} \cdot \mathbf{x}$, for all **x** in $\mathbf{R}^3$.

44. The cross product of two vectors in $\mathbf{R}^3$ is defined by $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$.

Consider an arbitrary vector **v** in $\mathbf{R}^3$. Is the transformation $T(\mathbf{x}) = \mathbf{v} \times \mathbf{x}$ from $\mathbf{R}^3$ to $\mathbf{R}^3$ linear? If so, find its matrix in terms of the components of the vector **v**.

Section 2.2:

4. Interpret the following linear transformation geometrically: $T(\mathbf{x}) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{x}$.

5. The matrix $\begin{bmatrix} -0.8 & -0.6 \\ 0.6 & -0.8 \end{bmatrix}$ represents a rotation. Find the angle of rotation (in radians).

6. Let L be the line in $\mathbf{R}^3$ that consists of all scalar multiples of the vector $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Find the orthogonal projection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto L.

7. Let L be the line in $\mathbf{R}^3$ that consists of all scalar multiples of the vector $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Find the reflection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ about the line L.

Find matrices of the linear transformations from $\mathbf{R}^3$ to $\mathbf{R}^3$ given in Exercises 19-23. Some of these transformations have not been formally defined in the text. Use common sense. You may assume that all these transformations are linear. [*Note*: Your answers to each of these problems should be a 3×3 matrix.]

19. The orthogonal projection onto the *xy*-plane.

20. The reflection about the *xz*-plane.

21. The rotation about the *z*-axis through an angle of $\pi/2$, counterclockwise as viewed from the positive *z*-axis.

22. The rotation about the *y*-axis through an angle θ, counterclockwise as viewed from the positive *y*-axis.

23. The reflection about the plane $y = z$.

34. One of the five given matrices represents an orthogonal projection onto a line and another represents a reflection about a line. Identify both and briefly justify your choice.

$$\mathbf{A} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \quad \mathbf{B} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \mathbf{C} = \frac{1}{3}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \mathbf{D} = -\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \quad \mathbf{E} = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

Section 2.3:

Decide whether the matrices in Exercises 2, 4,and 6 are invertible. If they are, find the inverse matrix. Do the computations with paper and pencil. Show all your work.

2. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ 4. $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ 6. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Section 2.4:

15. Compute the matrix product $\begin{bmatrix} 1 & -2 & -5 \\ -2 & 5 & 11 \end{bmatrix}\begin{bmatrix} 8 & -1 \\ 1 & 2 \\ 1 & -1 \end{bmatrix}$. Explain why the result does not contradict Fact 2.4.9.

[Fact 2.4.9 states that if **A** and **B** are two $n \times n$ matrices such that $\mathbf{BA} = \mathbf{I}_n$, then
(a) **A** and **B** are both invertible; (b) $\mathbf{A}^{-1} = \mathbf{B}$ and $\mathbf{B}^{-1} = \mathbf{A}$; and (c) $\mathbf{AB} = \mathbf{I}_n$.]

For two invertible $n \times n$ matrices **A** and **B**, determine which of the formulas stated in Exercises 16 through 25 are necessarily true.

16. $(\mathbf{I}_n - \mathbf{A})(\mathbf{I}_n + \mathbf{A}) = \mathbf{I}_n - \mathbf{A}^2$

17. $(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$

18. $\mathbf{A}^2$ is invertible, and $(\mathbf{A}^2)^{-1} = (\mathbf{A}^{-1})^2$

19. $\mathbf{A} + \mathbf{B}$ is invertible, and $(\mathbf{A} + \mathbf{B})^{-1} = \mathbf{A}^{-1} + \mathbf{B}^{-1}$

20. $(\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$

21. $\mathbf{ABB}^{-1}\mathbf{A}^{-1} = \mathbf{I}_n$

22. $\mathbf{ABA}^{-1} = \mathbf{B}$

23. $(\mathbf{ABA}^{-1})^3 = \mathbf{AB}^3\mathbf{A}^{-1}$

24. $(\mathbf{I}_n + \mathbf{A})(\mathbf{I}_n + \mathbf{A}^{-1}) = 2\mathbf{I}_n + \mathbf{A} + \mathbf{A}^{-1}$

25. $\mathbf{A}^{-1}\mathbf{B}$ is invertible, and $(\mathbf{A}^{-1}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}$

34. Consider two $n \times n$ matrices **A** and **B**, such that the product **AB** is invertible. Show that the matrices **A** and **B** are both invertible. *Hint*: $\mathbf{AB}(\mathbf{AB})^{-1} = \mathbf{I}_n$ and $(\mathbf{AB})^{-1}\mathbf{AB} = \mathbf{I}_n$. Use Fact 2.4.9.

44. Find all linear transformations T from $\mathbf{R}^2$ to $\mathbf{R}^2$ such that $T\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$ and $T\begin{bmatrix}2\\5\end{bmatrix}=\begin{bmatrix}1\\3\end{bmatrix}$.

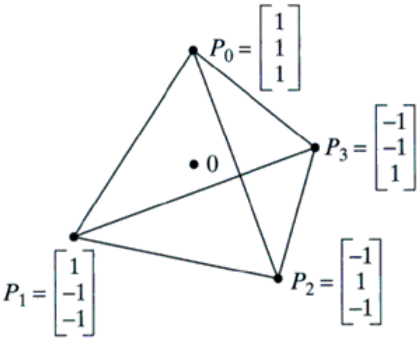
Hint: We are looking for the 2×2 matrices $\mathbf{A}$ such that $\mathbf{A}\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$ and $\mathbf{A}\begin{bmatrix}2\\5\end{bmatrix}=\begin{bmatrix}1\\3\end{bmatrix}$.

These two equations can be combined to form the matrix equation $\mathbf{A}\begin{bmatrix}1&2\\2&5\end{bmatrix}=\begin{bmatrix}2&1\\1&3\end{bmatrix}$.

46. Find the matrix of the linear transformation T from $\mathbf{R}^2$ to $\mathbf{R}^3$ with $T\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}7\\5\\3\end{bmatrix}$ and $T\begin{bmatrix}2\\5\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$.

(Compare with Exercise 45.)

48. Consider the regular tetrahedron sketched below, whose center is at the origin.



Let T from $\mathbf{R}^3$ to $\mathbf{R}^3$ be the rotation about the axis through points 0 and P_2 that transforms P_1 into P_3. Find the images of the four corners of the tetrahedron under this transformation.

$P_0 \xrightarrow{T}$

$P_1 \to P_3$

$P_2 \to$

$P_3 \to$

Let L from $\mathbf{R}^3$ to $\mathbf{R}^3$ be the reflection about the plane through the points 0, P_0, and P_3. Find the images of the four corners of the tetrahedron under this transformation.

$P_0 \xrightarrow{L}$

$P_1 \to$

$P_2 \to$

$P_3 \to$

Describe the transformations in parts (a) through (c) geometrically.

a. T^{-1} b. L^{-1} c. $T^2 = T \circ T$ (the composite of T with itself)

d. Find the images of the four corners under the transformations $T \circ L$ and $L \circ T$.
Are the two transformations the same?

$P_0 \xrightarrow{T\circ L}$	$P_0 \xrightarrow{L\circ T}$
$P_1 \to$	$P_1 \to$
$P_2 \to$	$P_2 \to$
$P_3 \to$	$P_3 \to$

e. Find the images of the four corners under the transformation $L \circ T \circ L$.
Describe this transformation geometrically.

49. Find the matrices of the transformations T and L defined in Exercise 48.

For additional practice (not to be turned in):
Section 2.1:

5. Consider the linear transformation T from $\mathbf{R}^3$ to $\mathbf{R}^2$ with $T\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}7\\11\end{bmatrix}$, $T\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}6\\9\end{bmatrix}$, and $T\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}-13\\17\end{bmatrix}$.

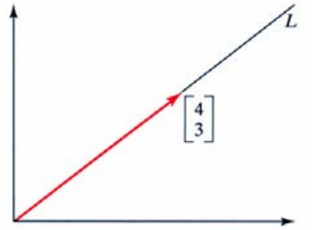
Find the matrix **A** of T.

9. Decide whether the matrix $\begin{bmatrix}2&3\\6&9\end{bmatrix}$ is invertible. Find the inverse if it exists.

Section 2.2:

10. Find the matrix of the projection onto the line L in $\mathbf{R}^2$ shown in the accompanying figure. [*Note*: Your answer should be a 2×2 matrix.]

11. Refer to Exercise 10. Find the matrix of the reflection about the line L. [*Note*: As in the previous problem, your answer should be a 2×2 matrix.]



Find matrices of the linear transformations from $\mathbf{R}^3$ to $\mathbf{R}^3$ given in Exercises 21 and 24. Some of these transformations have not been formally defined in the text. Use common sense. You may assume that all these transformations are linear.

27. Consider the matrices **A** through **E** below.

$$\mathbf{A}=\begin{bmatrix}0.6&0.8\\0.8&-0.6\end{bmatrix}\quad \mathbf{B}=\begin{bmatrix}3&0\\0&3\end{bmatrix}\quad \mathbf{C}=\begin{bmatrix}0.36&-0.48\\-0.48&0.64\end{bmatrix}\quad \mathbf{D}=\begin{bmatrix}-0.8&0.6\\-0.6&-0.8\end{bmatrix}\quad \mathbf{E}=\begin{bmatrix}1&0\\-1&1\end{bmatrix}$$

Fill in the blanks in the sentences below. We are told that there is a solution in each case.
Matrix ____ represents a scaling.
Matrix ____ represents a projection.
Matrix ____ represents a shear.
Matrix ____ represents a reflection.
Matrix ____ represents a rotation.

28. Each of the linear transformations in parts (a) through (e) corresponds to one (and only one) of the matrices **A** through **J**. Match them up.

a. Scaling
b. Shear
c. Rotation
d. Projection
e. Reflection

$$\mathbf{A}=\begin{bmatrix}0&0\\0&1\end{bmatrix}\quad \mathbf{B}=\begin{bmatrix}2&1\\1&0\end{bmatrix}\quad \mathbf{C}=\begin{bmatrix}-0.6&0.8\\-0.8&-0.6\end{bmatrix}\quad \mathbf{D}=\begin{bmatrix}7&0\\0&7\end{bmatrix}\quad \mathbf{E}=\begin{bmatrix}1&0\\-3&1\end{bmatrix}$$

$$\mathbf{F}=\begin{bmatrix}0.6&0.8\\0.8&-0.6\end{bmatrix}\quad \mathbf{G}=\begin{bmatrix}0.6&0.6\\0.8&0.8\end{bmatrix}\quad \mathbf{H}=\begin{bmatrix}2&-1\\1&2\end{bmatrix}\quad \mathbf{I}=\begin{bmatrix}0&0\\1&0\end{bmatrix}\quad \mathbf{J}=\begin{bmatrix}0.8&-0.6\\0.6&-0.8\end{bmatrix}$$

Section 2.3:

40. Show that if a square matrix **A** has two equal columns, then **A** is not invertible.

41. Which of the following linear transformations T from $\mathbf{R}^3$ to $\mathbf{R}^3$ are invertible? Find the inverse if it exists.
   a. Reflection about a plane.
   b. Projection onto a plane.
   c. Scaling by a factor of 5 [i.e., $T(\mathbf{v})=5\mathbf{v}$, for all vectors **v**].
   d. Rotation about an axis.

42. A square matrix is called a permutation matrix if it contains a 1 exactly once in each row and in each column, with all other entries being 0. Examples are the identity matrix $\mathbf{I}_n$ and $\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$. Are permutation matrices invertible? If so, is the inverse a permutation matrix as well?

43. Consider two invertible $n \times n$ matrices **A** and **B**. Is the linear transformation $\mathbf{y} = \mathbf{A}(\mathbf{B}\mathbf{x})$ invertible? If so, what is the inverse? [Hint: Solve the equation $\mathbf{y} = \mathbf{A}(\mathbf{B}\mathbf{x})$ first for $\mathbf{B}\mathbf{x}$ and then for **x**.]

53. Let $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix}$ in all parts of this problem.

a. Find a scalar λ (lambda) such that the matrix $\mathbf{A} - \lambda\mathbf{I}_2$ fails to be invertible. There are two solutions; choose one and use it in parts (b) and (c).

b. For the λ you chose in part (a), find the matrix $\mathbf{A} - \lambda\mathbf{I}_2$; then find a nonzero vector **x** such that $(\mathbf{A} - \lambda\mathbf{I}_2)\mathbf{x} = \mathbf{0}$. (This can be done, since $\mathbf{A} - \lambda\mathbf{I}_2$ fails to be invertible.)

c. Note that the equation $(\mathbf{A} - \lambda\mathbf{I}_2)\mathbf{x} = \mathbf{0}$ can be written as $\mathbf{A}\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}$, or, $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. Check that the equation $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ holds for your λ from part (a) and your **x** from part (b).

54. Let $\mathbf{A} = \begin{bmatrix} 1 & 10 \\ -3 & 12 \end{bmatrix}$. Using Exercise 53 (see below) as a guide, find a scalar λ and a nonzero vector **x** such that $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$.

Section 2.4:

If possible, compute the matrix products in Exercises 1-4, 7, and 10-12, using pencil and paper.

1. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 7 & 5 \\ 3 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

4. $\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

12. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

14. For the matrices $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{E} = \begin{bmatrix} 5 \end{bmatrix}$, determine which of the 25 matrix products **AA**, **AB**, **AC**, …, **ED**, **EE** are defined, and compute those that are defined.

28. Find a nonzero 2 × 2 matrix **A** such that $\mathbf{A}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

29. Find a nonzero 2 × 2 matrix **B** such that $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

31. For the matrix $\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, find a matrix **A** such that $\mathbf{B}\mathbf{A} = \mathbf{I}_2$.
How many solutions **A** does this problem have?

32. Can you find a 3 × 2 matrix **A** and a 2 × 3 matrix **B** such that such that the product **AB** is $\mathbf{I}_3$?
Hint: Consider any 3 × 2 matrix **A** and any 2 × 3 matrix **B**. There is a nonzero vector **x** in $\mathbf{R}^3$ such that $\mathbf{B}\mathbf{x} = \mathbf{0}$. (Why?) Now think about **ABx**.

33. Can you find a 3 × 2 matrix **A** and a 2 × 3 matrix **B** such that such that the product **AB** is invertible? (The hint in Exercise 32 applies.)

36. Find all 2 × 2 matrices **X** such that **AX** = **B**, where $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

63. Prove the *distributive laws* for matrices: $\mathbf{A}(\mathbf{C} + \mathbf{D}) = \mathbf{AC} + \mathbf{AD}$ and $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$.

Extra problems for those interested in economics:

Section 2.3:

49. ***Input-Output Analysis***. (This exercise builds on Exercises 1.1.20, 1.2.37, 1.2.38, and 1.2.39.) Consider the industries $J_1, J_2, \ldots, J_n$ in an economy. Suppose the consumer demand vector is **b**, the output vector is **x** and the demand of the jth industry is $\mathbf{v}_j$. (The ith component a_{ij} of $\mathbf{v}_j$ is the demand industry J_j puts on industry J_i, per unit of output of J_j.) As we have seen in Exercise 1.2.38, the output **x** just meets the aggregate demand if

$$\underbrace{x_1\mathbf{v}_1 + x_2\mathbf{v}_2 \cdots x_n\mathbf{v}_n + \mathbf{b}}_{\text{aggregate demand}} = \underbrace{\mathbf{x}}_{\text{output}} .$$

This equation can be written more succinctly as

$$\begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \mathbf{b} = \mathbf{x}$$

or $\mathbf{Ax} + \mathbf{b} = \mathbf{x}$. The matrix **A** is called the *technology matrix* of this economy; its coefficients a_{ij} describe the interindustry demand, which depend on the technology used in the production process. The equation

$$\mathbf{Ax} + \mathbf{b} = \mathbf{x}$$

describes a linear system, which we can write in the customary form:

$$\mathbf{x} - \mathbf{Ax} = \mathbf{b}$$
$$\mathbf{I}_n\mathbf{x} - \mathbf{Ax} = \mathbf{b}$$
$$(\mathbf{I}_n - \mathbf{A})\mathbf{x} = \mathbf{b}$$

If we want to know the output **x** required to satisfy a given consumer demand **b** (this was our objective in the previous exercises), we can solve this linear system, preferably via the augmented matrix.

In economics, however, we often ask the other questions: If **b** changes, how will **x** change in response. If the consumer demand on one industry increases by 1 unit and the consumer demand on the other industries remains unchanged, how will **x** change? If we ask questions like these, we think of the output **x** as a *function* of the consumer demand **b**.

If the matrix $(\mathbf{I}_n - \mathbf{A})$ is invertible, we can express **x** as a function **b** (in fact, as a linear transformation):

$$\mathbf{x} = (\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{b}$$

a. Consider the economy of Israel in 1958 (discussed in Exercise 1.2.39). Find the technology matrix **A**, the matrix $(\mathbf{I}_n - \mathbf{A})$, and its inverse $(\mathbf{I}_n - \mathbf{A})^{-1}$.

b. In the example discussed in part (a), suppose the consumer demand on agriculture (Industry 1) is 1 unit (1 million pounds), and the demands on the other two industries are zero. What output **x** is required in this case? How does your answer relate to the matrix $(\mathbf{I}_n - \mathbf{A})^{-1}$?

c. Explain, in terms of economics, why the diagonal elements of the matrix $(\mathbf{I}_n - \mathbf{A})^{-1}$ you found in part (a) must be at least 1.

d. If the consumer demand on manufacturing increases by 1 (from whatever it was), and the consumer demand on the other two industries remains the same, how will the output have to change? How does your answer relate to the matrix $(\mathbf{I}_n - \mathbf{A})^{-1}$?

e. Using your answers in parts (a) through (d) as a guide, explain in general (not just for this example) what the columns and the entries of the matrix $(\mathbf{I}_n - \mathbf{A})^{-1}$ tell you, in terms of economics. Those who have studied multivariable calculus may wish to consider the partial derivatives $\dfrac{\partial x_i}{\partial b_j}$.

50. This exercise refers to Exercise 49a. Consider the entry $k = a_{11} = 0.293$ of the technology matrix **A**. Verify that the entry in the first row and the first column of $(\mathbf{I}_n - \mathbf{A})^{-1}$ is the value of the geometric series $1 + k + k^2 + \cdots$. Interpret this observation in terms of economics.

TRUE OR FALSE?

1. The function $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-y \\ y-x \end{bmatrix}$ is a linear transformation.

2. Matrix $\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$ represents a rotation.

3. If A is any invertible $n \times n$ matrix, then $\operatorname{rref}(A) = I_n$.

4. The formula $(A^2)^{-1} = (A^{-1})^2$ holds for all invertible matrices A.

5. The formula $AB = BA$ holds for all $n \times n$ matrices A and B.

6. If $AB = I_n$ for two $n \times n$ matrices A and B, then A must be the inverse of B.

7. If A is a 3×4 matrix and B is a 4×5 matrix, then AB will be a 5×3 matrix.

8. The function $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 1 \end{bmatrix}$ is a linear transformation.

9. The matrix $\begin{bmatrix} 5 & 6 \\ -6 & 5 \end{bmatrix}$ represents a rotation combined with a scaling.

10. If A is any invertible $n \times n$ matrix, then A commutes with A^{-1}.

11. Matrix $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is invertible.

12. Matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is invertible.

13. There exists an upper triangular 2×2 matrix A such that $A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

14. The function $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (y+1)^2 - (y-1)^2 \\ (x-3)^2 - (x+3)^2 \end{bmatrix}$ is a linear transformation.

15. Matrix $\begin{bmatrix} k & -2 \\ 5 & k-6 \end{bmatrix}$ is invertible for all real numbers k.

16. There exists a real number k such that the matrix $\begin{bmatrix} k-1 & -2 \\ -4 & k-3 \end{bmatrix}$ fails to be invertible.

17. There exists a real number k such that the matrix $\begin{bmatrix} k-2 & 3 \\ -3 & k-2 \end{bmatrix}$ fails to be invertible.

18. Matrix $\begin{bmatrix} -0.6 & 0.8 \\ -0.8 & -0.6 \end{bmatrix}$ represents a rotation.

19. The formula $\det(2A) = 2\det(A)$ holds for all 2×2 matrices A.

20. There exists a matrix A such that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

21. There exists a matrix A such that $A\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$.

22. There exists a matrix A such that $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

23. The matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ represents a reflection about a line.

24. $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 3k \\ 0 & 1 \end{bmatrix}$ for all real numbers k.

25. The matrix product $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ is always a scalar multiple of I_2.

26. There exists a nonzero upper triangular 2×2 matrix A such that $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

27. There exists a positive integer n such that $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^n = I_2$.

28. There exists an invertible 2×2 matrix A such that $A^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

29. There exists an invertible $n \times n$ matrix with two identical rows.

30. If $A^2 = I_n$, then matrix A must be invertible.

31. If $A^{17} = I_2$, then matrix A must be I_2.

32. If $A^2 = I_2$, then matrix A must be either I_2 or $-I_2$.

33. If matrix A is invertible, then matrix $5A$ must be invertible as well.

34. If A and B are two 4×3 matrices such that $A\vec{v} = B\vec{v}$ for all vectors $\vec{v}$ in $\mathbb{R}^3$, then matrices A and B must be equal.

35. If matrices A and B commute, then the formula $A^2B = BA^2$ must hold.

36. If $A^2 = A$ for an invertible $n \times n$ matrix A, then A must be I_n.

37. If matrices A and B are both invertible, then matrix $A+B$ must be invertible as well.

38. The equation $A^2 = A$ holds for all 2×2 matrices A representing a projection.

39. If matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is invertible, then matrix $\begin{bmatrix} a & b \\ d & e \end{bmatrix}$ must be invertible as well.

40. If A^2 is invertible, then matrix A itself must be invertible.

41. The equation $A^{-1} = A$ holds for all 2×2 matrices A representing a reflection.

42. The formula $(A\vec{v}) \cdot (A\vec{w}) = \vec{v} \cdot \vec{w}$ holds for all invertible 2×2 matrices A and for all vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^2$.

43. There exist a 2×3 matrix A and a 3×2 matrix B such that $AB = I_2$.

44. There exist a 3×2 matrix A and a 2×3 matrix B such that $AB = I_3$.

45. If $A^2 + 3A + 4I_3 = 0$ for a 3×3 matrix A, then A must be invertible.

46. If A is an $n \times n$ matrix such that $A^2 = 0$, then matrix $I_n + A$ must be invertible.

47. If matrix A commutes with B, and B commutes with C, then matrix A must commute with C.

48. If T is any linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$, then $T(\vec{v} \times \vec{w}) = T(\vec{v}) \times T(\vec{w})$ for all vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$.

49. There exists an invertible 10×10 matrix that has 92 ones among its entries.

50. The formula $\operatorname{rref}(AB) = \operatorname{rref}(A)\operatorname{rref}(B)$ holds for all $n \times p$ matrices A and for all $p \times m$ matrices B.

51. There exists an invertible matrix S such that $S^{-1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S$ is a diagonal matrix.

52. If the linear system $A^2\vec{x} = \vec{b}$ is consistent, then the system $A\vec{x} = \vec{b}$ must be consistent as well.

53. There exists an invertible 2×2 matrix A such that $A^{-1} = -A$.

54. There exists an invertible 2×2 matrix A such that $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

55. If a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ represents the orthogonal projection onto a line L, then the equation $a^2 + b^2 + c^2 + d^2 = 1$ must hold.

56. If A is an invertible 2×2 matrix and B is any 2×2 matrix, then the formula $\operatorname{rref}(AB) = \operatorname{rref}(B)$ must hold.