Math 15a – Fall 2007 – Homework #1

Problems due Mon, Sept 10:

Section 1.1:

11. Find all solutions of the linear system $\begin{Bmatrix} x-2y=2 \\ 3x+5y=17 \end{Bmatrix}$. Represent your solutions graphically, as intersections of lines in the *xy*-plane.

16. Find all solutions of the linear system $\begin{Bmatrix} x+4y+z=0 \\ 4x+13y+7z=0 \\ 7x+22y+13z=0 \end{Bmatrix}$. Describe your solution in terms of intersecting planes. You need not sketch these planes.

Note: In understanding the relationship between linear equations and the geometry of planes in $\mathbf{R}^3$, it is helpful to understand the *dot product*, a topic covered in Multivariable Calculus. There's a brief summary of vectors, the dot product and the cross product in Appendix A of the Bretscher text, but it's best to consult any standard multivariable calculus text for a more complete treatment of these topics. A **supplement** on this topic is posted on the course website. We can also go over this in more detail outside of class.

17. Find all solutions of the linear system $\begin{Bmatrix} x+2y=a \\ 3x+5y=b \end{Bmatrix}$, where *a* and *b* are arbitrary constants.

25. Consider the linear system $\begin{Bmatrix} x+y-z=-2 \\ 3x-5y+13z=18 \\ x-2y+5z=k \end{Bmatrix}$, where *k* is an arbitrary number.

   a. For which value(s) of *k* does this system have one or infinitely many solutions?
   b. For each value of *k* you found in part a, how many solutions does the system have?
   c. Find all solutions for each value of *k*.

29. Find the polynomial of degree 2 [a polynomial of the form $f(t)=a+bt+ct^2$] whose graph goes through the points $(1,-1)$, $(2,3)$, and $(3,13)$. Sketch the graph of this polynomial.

Section 1.2:

In exercises 9 and 10, find all solutions of the equations with paper and pencil using Gauss-Jordan elimination. Show all your work.

9. $\begin{Bmatrix} x_4+2x_5-x_6=2 \\ x_1+2x_2+x_5-x_6=0 \\ x_1+2x_2+2x_3-x_5+x_6=2 \end{Bmatrix}$

10. $\begin{Bmatrix} 4x_1+3x_2+2x_3-x_4=4 \\ 5x_1+4x_2+3x_3-x_4=4 \\ -2x_1-2x_2-x_3+2x_4=-3 \\ 11x_1+6x_2+4x_3+x_4=11 \end{Bmatrix}$

16. Solve the linear system $\begin{Bmatrix} 3x_1+6x_2+9x_3+5x_4+25x_5=53 \\ 7x_1+14x_2+21x_3+9x_4+53x_5=105 \\ -4x_1-8x_2-12x_3+5x_4-10x_5=11 \end{Bmatrix}$. You may use technology.

30. Find the polynomial of degree 3 [a polynomial of the form $f(t)=a+bt+ct^2+dt^3$] whose graph goes through the points $(0,1)$, $(1,0)$, $(-1,0)$, and $(2,-15)$. Sketch the graph of this cubic.

34. The dot product of two vectors $\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{y}=\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ in $\mathbf{R}^n$ is defined by $\mathbf{x}\cdot\mathbf{y}=x_1y_1+x_2y_2+\cdots+x_ny_n$.

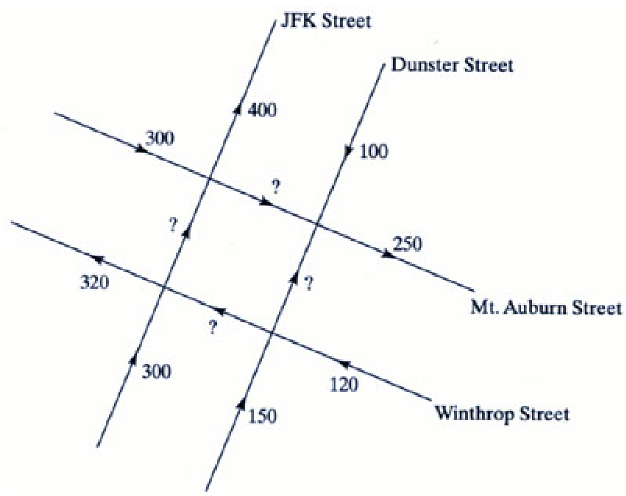
    Note that the dot product of two vectors is a scalar. We say that the vectors **x** and **y** are *perpendicular* if $\mathbf{x}\cdot\mathbf{y}=0$. Find all vectors in $\mathbf{R}^3$ perpendicular to $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$.

36. Find all solutions x_1, x_2, x_3 of the equation $\mathbf{b} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3$, where

$$\mathbf{b} = \begin{bmatrix} -8 \\ -1 \\ 2 \\ 15 \end{bmatrix},\ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \\ 5 \end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \\ 3 \end{bmatrix},\ \mathbf{v}_3 = \begin{bmatrix} 4 \\ 6 \\ 9 \\ 1 \end{bmatrix}.$$

42. The accompanying sketch represents a maze of one-way streets in a city in the United States. The traffic volume through certain blocks during an hour has been measured. Suppose that the vehicles leaving the area during this hour were exactly the same as those entering it.

What can you say about the traffic volume at the four locations indicated by a question mark? Can you figure out exactly how much traffic there was on each block? If not, describe one possible scenario. For each of the four locations, find the highest and lowest possible traffic volume.



60. "A rooster is worth five coins, a hen three coins, and 3 chicks one coin. With 100 coins we buy 100 of them. How many roosters, hens, and chicks can we buy?"
(From the *Mathematical Manual* by Zhang Qiujian, Ch. 3, Prob. 38; 5th century A.D.)

Section 1.3:

Find the rank of the matrices in Exercises 2 through 4: 2. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ 3. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ 4. $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

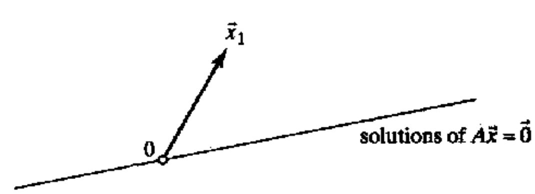
24. Let **A** be a 4 × 4 matrix, and let **b** and **c** be two vectors in $\mathbf{R}^4$. We are told that the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution. What can you say about the number of solutions of the system $\mathbf{Ax} = \mathbf{c}$?

26. Let **A** be a 4 × 3 matrix, and let **b** and **c** be two vectors in $\mathbf{R}^4$. We are told that the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution. What can you say about the number of solutions of the system $\mathbf{Ax} = \mathbf{c}$?

47. A linear system of the form $\mathbf{Ax} = \mathbf{0}$ is called *homogeneous*. [Matrices and vectors are indicated in **bold**.] Justify the following facts:
a. All homogeneous systems are consistent.
b. A homogeneous system with fewer equations than unknowns has infinitely many solutions.
c. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of the homogeneous system $\mathbf{Ax} = \mathbf{0}$, then $\mathbf{x}_1 + \mathbf{x}_2$ is a solution as well.
d. If $\mathbf{x}$ is a solution of the homogeneous system $\mathbf{Ax} = \mathbf{0}$ and if *k* is an arbitrary constant, then $k\mathbf{x}$ is a solution as well.

48. Consider a solution $\mathbf{x}_1$ of the linear system $\mathbf{Ax} = \mathbf{b}$. Justify the facts stated in parts (a) and (b):
a. If $\mathbf{x}_h$ is a solution of the system $\mathbf{Ax} = \mathbf{0}$, then $\mathbf{x}_1 + \mathbf{x}_h$ is a solution of the system $\mathbf{Ax} = \mathbf{b}$.
b. If $\mathbf{x}_2$ is another solution of the system $\mathbf{Ax} = \mathbf{b}$, then $\mathbf{x}_2 - \mathbf{x}_1$ is a solution of the system $\mathbf{Ax} = \mathbf{0}$.
c. Now suppose **A** is a 2 × 2 matrix. A solution vector $\mathbf{x}_1$ of the system $\mathbf{Ax} = \mathbf{b}$ is shown in the accompanying figure. We are told that the solutions of the system $\mathbf{Ax} = \mathbf{0}$ form the line shown in the sketch. Draw the line consisting of all solutions of the system $\mathbf{Ax} = \mathbf{b}$.

If you are puzzled by the generality of this problem, think about an example first: $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$, and $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



For additional practice:

Section 1.1:

In Exercises 1, 3, and 7, find all solutions of the linear systems using elimination (as discussed in this section). Then check your solutions.

1. $\begin{Bmatrix} x+2y=1 \\ 2x+3y=1 \end{Bmatrix}$ 3. $\begin{Bmatrix} 2x+4y=3 \\ 3x+6y=2 \end{Bmatrix}$ 7. $\begin{Bmatrix} x+2y+3z=1 \\ x+3y+4z=3 \\ x+4y+5z=4 \end{Bmatrix}$

15. Find all solutions of the linear system $\begin{Bmatrix} x+y-z=0 \\ 4x-y+5z=0 \\ 6x+y+4z=0 \end{Bmatrix}$. Describe your solutions in terms of intersecting planes. You need not sketch these planes.

Section 1.2:

In exercises 5 and 11, find all solutions of the equations with paper and pencil using Gauss-Jordan elimination. Show all your work.

5. $\begin{Bmatrix} x_3+x_4=0 \\ x_2+x_3=0 \\ x_1+x_2=0 \\ x_1+x_4=0 \end{Bmatrix}$ 11. $\begin{Bmatrix} x_1+2x_3+4x_4=-8 \\ x_2-3x_3-x_4=6 \\ 3x_1+4x_2-6x_3+8x_4=0 \\ -x_2+3x_3+4x_4=-12 \end{Bmatrix}$

20. We say that two $n \times m$ matrices in reduced row-echelon form are of the same type if they contain the same number of leading 1's in the same positions. For example,

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are of the same type. How many types of 2×2 matrices in reduced row-echelon form are there?

21. How many types of 3×2 matrices in reduced row-echelon form are there? (See Exercise 20.)

22. How many types of 2×3 matrices in reduced row-echelon form are there? (See Exercise 20.)

Section 1.3:

22. Consider a linear system of three equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix look like? Explain your answer.

23. Consider a linear system of four equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix look like? Explain your answer.

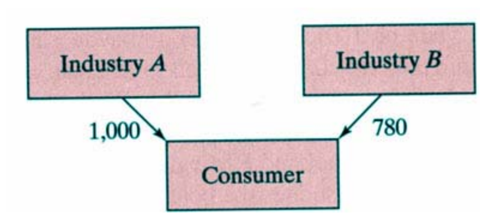
Extra problems for those interested in economics:

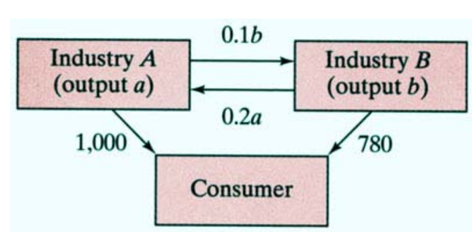
Section 1.1:

20. The Russian-born U.S. economist and Nobel laureate Wassily Leontief (1906-1999) was interested in the following question: What output should each of the industries in an economy produce to satisfy the total demand for all products? Here, we consider a very simple example of input-output analysis, an economy with only two industries, *A* and *B*. Assume that the consumer demand for their products is, respectively, 1000 and 780, in millions of dollars per year.

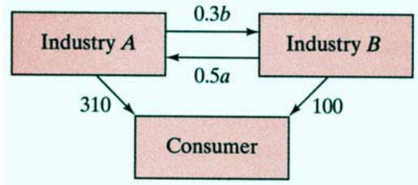
What outputs *a* and *b* (in millions of dollars per year) should the two industries generate to satisfy the demand?

You may be tempted to say 1000 and 780, respectively, but things are not quite as simple as that. We have to take into account the interindustry demand as well. Let us say that industry A produces electricity. Of course, producing almost any product will require electric power. Suppose that industry B needs 10¢ worth of electricity for each \$1 of output B produces and that industry A needs 20¢ worth of B's products for each \$1 of output A produces. Find the outputs *a* and *b* needed to satisfy both consumer and interindustry demand.



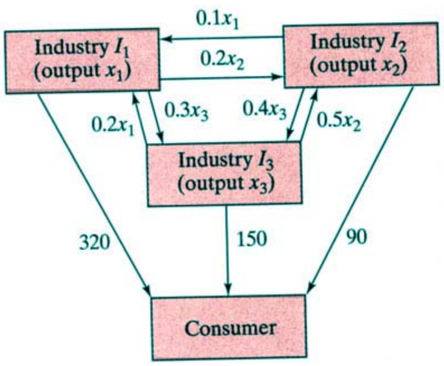


21. Find the outputs a and b needed to satisfy the consumer and interindustry demands given in the following figure (see Exercise 20.):



Section 1.2:

37. For some background on this exercise, see Exercise 1.1.20. Consider an economy with three industries, I_1, I_2, I_3. What outputs x_1, x_2, and x_3 should they produce to satisfy both consumer demand and interindustry demand? The demands put on the three industries are shown in the accompanying figure.



38. If we consider more than three industries in an input-output model, it is cumbersome to represent all the demands in a diagram as in Exercise 37. Suppose we have industries $I_1, I_2, \ldots, I_n$ with outputs $x_1, x_2, \ldots, x_n$. The *output vector* is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. The *consumer demand* vector is $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$, where b_i is the consumer demand on industry I_i. The demand vector for industry I_j is $\mathbf{v}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}$ where a_{ij} is the demand industry I_j puts on industry I_i, for each \$1 of output industry I_j produces. For example, $a_{32} = 0.5$ means that industry I_2 needs 50¢ worth of products from industry I_3 for each \$1 of goods I_2 produces. The coefficient a_{ii} need not be 0: Producing a product may require goods or services from the same industry.

a. Find the four demand vectors for the economy in Exercise 37.

b. What is the meaning in economic terms of $x_j \mathbf{v}_j$?

c. What is the meaning in economic terms of $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n + \mathbf{b}$?

d. What is the meaning in economic terms of the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n + \mathbf{b} = \mathbf{x}$?

39. Consider the economy of Israel in 1958. [Ref.: W. Leontief: Input-Output Economics, Oxford University Press, 1966.] The three industries considered here are

$$I_1: \text{ agriculture;}$$
$$I_2: \text{ manufacturing;}$$
$$I_3: \text{ energy.}$$

Outputs and demands are measured in millions of Israeli pounds, the currency of Israel at that time.

We are told that $\mathbf{b} = \begin{bmatrix} 13.2 \\ 17.6 \\ 1.8 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 0.293 \\ 0.014 \\ 0.044 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0.207 \\ 0.01 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0.017 \\ 0.216 \end{bmatrix}$.

a. Why do the first components of $\mathbf{v}_2$ and $\mathbf{v}_3$ equal 0?

b. Find the outputs x_1, x_2, and x_3 required to satisfy demand.